8

Integral Transforms Related to the Fourier Transform

Fourier analysis can take different forms as we adapt it to various problems at hand. The main results of the Fourier integral theorem are used to justify continuous partial-wave analyses in terms of functions other than the oscillating exponential ones. Section 8.1 presents the bilateral and the more common unilateral Laplace transform where the expanding functions are the decreasing exponential functions exp(pq). In Section 8.2 we expand functions in terms of powers $x_{\pm}^{iq-1/2}$ (bilateral Mellin) or of powers x^q (common Mellin) as a continuous analogue of the ordinary Taylor series expansion. Section 8.3 deals with Fourier transforms of functions of N variables and applies them to the general solution of the N-dimensional elastic–diffusive equation. In particular, the three-dimensional wave and general heat equations are treated. Hankel transforms (Section 8.4) use the Bessel functions as the expanding set and arise out of N-dimensional Fourier transforms of functions of the radius. The elastic–diffusive equation solutions are completed, and the difference between odd and even dimensions is pointed out. We list, finally, several transform pairs which use cylindrical functions as their expanding set. Under the title of "other" integral transforms, in Section 8.5 we give a rough outline of the Sturm–Liouville approach. This is applied in particular to transforms using Airy functions. Other approaches lead to Hilbert and Stieltjes transforms. All sections are basically independent of one another except for Hankel transforms, which are built out of N-dimensional Fourier transforms. Those transforms which are only briefly mentioned in the text are accompanied by a bibliographical survey.

8.1. Laplace Transforms

The Laplace transform is essentially the Fourier transform on the imaginary axis of the transform argument. The direct implementation of this idea leads to the *bilateral* Laplace transform. The more commonly known version of this transform, the *unilateral* Laplace transform, is obtained for causal functions, i.e., those which are zero on the negative half-line. The Laplace transform, formulated in this way, allows in a rather natural way for the introduction of the initial conditions in the solution of certain differential equations.

8.1.1. Bilateral Laplace Transforms

Consider the Fourier transform pair, Eqs. (7.1). By setting $p =: -is$, the pair now reads

$$(\mathbb{L}_B^{-1}\mathbf{f}^{BL})(q) := f(q) = -i(2\pi)^{-1/2}\int_{-i\infty}^{i\infty} ds f^{BL}(s)\exp(qs), \qquad (8.1a)$$

$$(\mathbb{L}_B\mathbf{f})(s) := f^{BL}(s) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} dq f(q)\exp(-qs), \qquad (8.1b)$$

where we have also put $f^{BL}(s) := \tilde{f}(-is)$, thus defining $f^{BL}(s)$ as the *bilateral Laplace transform* of $f(q)$. In terms of the new transform functions, the Parseval identity appears as

$$\int_{-\infty}^{\infty} dq f(q)^* g(q) = -i\int_{-i\infty}^{i\infty} ds f^{BL}(s)^* g^{BL}(s). \qquad (8.1c)$$

We note that not every function which has a Fourier transform is bound to have a Laplace transform as the integral (8.1b) may well diverge. This happens when $f(q)$ behaves asymptotically like any finite negative power of q since the exponential kernel dominates the growth of the integrand.

8.1.2. Exponential Growth

To describe the regions in the s-plane where (8.1b) converges, it is convenient to introduce definitions concerning asymptotic exponential growth. If there exist constants k', n', and c' such that for $|q| \to \infty$

$$|f(q)| < k'\exp(c'|q|^{n'}) \qquad (8.2)$$

and if n and c are the minima of the n' and c' for which (8.2) holds, $f(q)$ is said to be of *order* n, *type* c, and *growth* (n, c). When we consider q real, we may examine separately the cases $q \to +\infty$ and $q \to -\infty$.

The growth of a Gaussian $G_\omega(q) \sim \exp(-q^2/2\omega)$ is thus $(2, -\mathrm{Re}(1/2\omega))$. A simple exponential function $\exp(aq)$, $a \in \mathscr{C}$, will be of growth $(1, \mathrm{Re}\, a)$ for $q > 0$ and $(1, -\mathrm{Re}\, a)$ for $q < 0$. Constants are of growth $(0, c)$. If two functions $f_1(q)$ and $f_2(q)$ are of the same order n, the type of their product is the sum of their types: $c_1 + c_2$. If their order is different, say $n_1 > n_2$, the growth of their product is (n_1, c_1).

For the bilateral Laplace transform of a (locally integrable) function $f(q)$ to exist, it is sufficient that the integrand in (8.1b) be of growth $(n > 0, c < 0)$. Due to the factor $\exp(-qs)$ of growth $(1, \mp s)$ for $q \gtrless 0$, $s \in \mathscr{R}$, we can contemplate three cases for the growth (n_f^+, c_f^+) of $f(q)$ at $q \to +\infty$:

(a) If $n_f < 1$, the growth of the integrand will be determined by that of the exponential factor, which is $(1, -s)$.
(b) If $n_f = 1$, the integrand growth will be $(1, c_f^+ - s)$.
(c) If $n_f > 1$, the growth will be (n_f^+, c_f^+).

At $q \to -\infty$, let the growth of $f(q)$ be (n_f^-, c_f^-). In the three cases the integrand growth will be as follows:

(a′) $(1, s)$ for $n_f^- < 1$;
(b′) $(1, c_f^- + s)$ for $n_f^- = 1$;
(c′) (n_f^-, c_f^-) for $n_f^- > 1$.

The $q > 0$ part allows integration for (a) $\operatorname{Re} s > 0$, (b) $\operatorname{Re} s > c_f^+$, and (c) all s as long as $c_f^+ < 0$. The $q < 0$ part, independently, allows integration for (a′) $\operatorname{Re} s > 0$, (b′) $\operatorname{Re} s < -c_f^-$, and (c′) all s as long as $c_f^- < 0$. The bilateral Laplace transform $f^{BL}(s)$ will thus exist for some region in the complex s-plane only if both parts permit integration. Conditions (a) and (b) restrict the allowed region to a right half-plane, while (a′) and (b′) restrict it to a left half-plane (Fig. 8.1). Only if the two half-planes have a nonempty

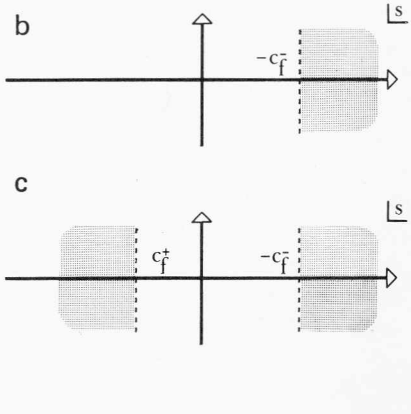


Fig. 8.1. Restrictions on the values of the complex variable s where the bilateral Laplace transform (8.1b) exists. (a) For $q > 0$; (b) for $q < 0$. Shaded areas indicate that the integral diverges. For the overlap between the two allowed regions (c), the transform exists.

overlap band will $f^{BL}(s)$ exist, and then only within this band. Cases (c) and (c′), when allowed, impose no restrictions on the s region.

8.1.3. Examples

As an example, consider $E_{\|}(q) := \exp(c|q|)$, $c \in \mathscr{C}$. The growth of this function in both directions is (1, Re c); hence the bilateral Laplace integral (8.1b) exists, according to cases (b′) and (b), for $-\operatorname{Re} c < \operatorname{Re} s < \operatorname{Re} c$, i.e., a vertical band in the complex s-plane centered about the imaginary axis. It is

$$\begin{aligned} E_{\|}^{BL}(s) &= (2\pi)^{-1/2}\left\{\int_{-\infty}^{0} dq \exp[-q(s-c)] + \int_{0}^{\infty} dq \exp[-q(s+c)]\right\} \\ &= (2\pi)^{-1/2}[1/(s+c) - 1/(s-c)], \qquad -\operatorname{Re} c < \operatorname{Re} s < \operatorname{Re} c. \end{aligned} \tag{8.3}$$

The transform function $E_{\|}^{BL}(s)$ is thus seen to have poles at $s = c$ and $s = -c$ which lie on the boundary of the existence band and determine its width. Perhaps surprisingly, the function (8.3) appears to be a well-defined function throughout the complex s-plane. Beyond the band boundaries, this *analytic continuation* of $E_{\|}^{BL}(s)$ is *not* the bilaterial Laplace transform of any function. Yet it can be used for contour integration purposes. Consider the task of finding the *inverse* transform (8.1a) of (8.3): Fig. 8.2. The integral

$$-i(2\pi)^{-1}\int_{-i\infty}^{i\infty} ds[1/(s+c) - 1/(s-c)]\exp(qs) \tag{8.4}$$

can be found for $q > 0$ by closing the integration contour counterclockwise in the Re $s < 0$ half-plane and using the familiar Cauchy and Jordan results.

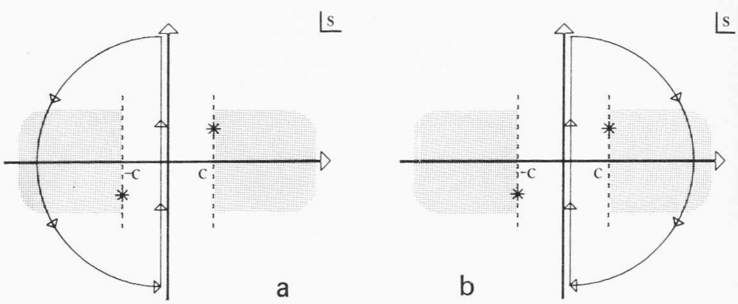


Fig. 8.2. Regions of existence of the bilateral Laplace transform (8.3) (unshaded). Asterisks indicate the locations of the poles of the function. On calculating the inverse Laplace transform (8.4), the integration can be performed for (a) $q > 0$ and (b) $q < 0$ using the analytic continuation of the function and complex contour integration techniques.

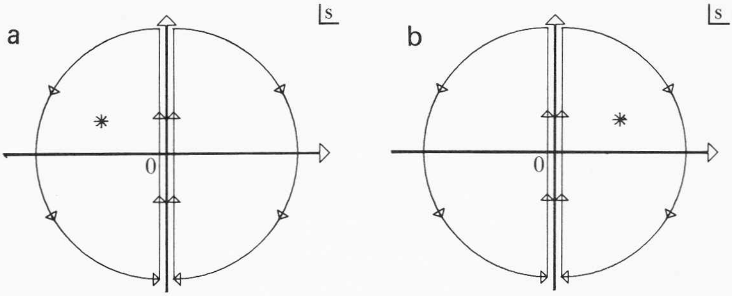


Fig. 8.3. Complex integration contours used in calculating the inverse bilateral Laplace transform in Eq. (8.6) for (a) Re $c < 0$ and (b) Re $c > 0$.

As the contour will enclose the $s = -c$ pole of residue $\exp[q(-c)]$, the result is $\exp(-qc)$. When $q < 0$, the contour may be closed in the Re $s > 0$ half-plane, the result being, as expected, $\exp(qc)$. The reconstitution of the original function for $q \in \mathscr{R}$ is thus $\exp(c|q|)$.

The convergence band in the s-plane must, however, be specified. It is part of the definition of the transform function. To illustrate this, consider the two functions

$$E_+(q) := \begin{cases} \exp(cq), & q > 0, \\ 0, & q \leqslant 0, \end{cases} \qquad E_-(q) := \begin{cases} 0, & q \geqslant 0, \\ \exp(cq), & q < 0. \end{cases} \tag{8.5}$$

Performing the integration in (8.1b), we see that

$$\begin{aligned} E_+^{BL}(s) &= (2\pi)^{-1/2}(s - c)^{-1}, \qquad \operatorname{Re} s > \operatorname{Re} c; \\ E_-^{BL}(s) &= -(2\pi)^{-1/2}(s - c)^{-1}, \qquad \operatorname{Re} s < \operatorname{Re} c. \end{aligned} \tag{8.6}$$

If we are asked to perform the inverse bilateral Laplace transform (8.1a) of $(2\pi)^{-1/2}(s - c)^{-1}$—without specifying the existence region—following the usual complex contour integration techniques (Fig. 8.3), we would come up with $E_+(q)$ if Re $c < 0$ or with $E_-(q)$ if Re $c > 0$.

Exercise 8.1. Show that the preceding paradox is resolved when we note that in taking the bilateral Laplace transform of $E_+(q)$ for Re $c > 0$ or that of $E_-(q)$ for Re $c < 0$ we are actually violating the conditions of the Fourier integral theorem. The inverse transform integrates them over the nonexistence region of the functions.

8.1.4. Unilateral Laplace Transforms

It might appear that the change of variables $p =: -is$ involved in defining the bilateral Laplace transform out of the Fourier transform has little new to offer us in the way of techniques for solving problems which do not yield to the Fourier methods. The "paradox" involved in (8.5)–(8.6), however, suggests a fruitful restatement of the transform which makes it applicable to time evolution of systems by *causal* functions, i.e., functions which are zero for negative values of the argument. The study of these functions and the behavior of their Fourier transforms in the complex plane occupied Section 7.4. Some significant computational and conceptual simplifications are obtained by the Laplace transform method. It will allow us to find solutions $f(q)$ to differential equations which can exhibit exponential growth $(1, c)$ for any finite c—that is, oscillating, damped, or exponentially growing solutions—in terms of the initial conditions at $q = 0$: $f(0)$, $df(q)/dq|_{q=0}$, and/or higher derivatives according to the order of the differential equation.

Let $f(q)$ be a function of growth $(1, c)$ in the positive q direction and let $\gamma > \operatorname{Re} c$. Build the function

$$f_\gamma(q) := \begin{cases} \exp(-\gamma q) f(q), & q > 0, \\ f(0)/2, & q = 0, \\ 0, & q < 0, \end{cases} \tag{8.7}$$

which is absolutely integrable. Assuming the other conditions of the Fourier integral theorem hold, the Fourier transform of (8.7) is

$$\tilde{f}_\gamma(p) = (2\pi)^{-1/2} \int_0^\infty dq f(q) \exp[-q(\gamma + ip)]. \tag{8.8}$$

We now perform the change of variable $s := \gamma + ip$ and set $f^L(s) := (2\pi)^{1/2} \tilde{f}_\gamma(p)$—the constant $(2\pi)^{1/2}$ is introduced so as to conform to custom. The Fourier transform pair (7.1) thus becomes

$$(\mathbb{L}^{-1}\mathbf{f}^L)(q) := f(q) = (2\pi i)^{-1} \int_{\gamma - i\infty}^{\gamma + i\infty} ds f^L(s) \exp(qs), \qquad q > 0, \tag{8.9a}$$

$$(\mathbb{L}\mathbf{f})(s) := f^L(s) = \int_0^\infty dq f(q) \exp(-qs), \qquad \operatorname{Re} s > \operatorname{Re}(\text{type } f). \tag{8.9b}$$

The function $f^L(s)$ is said to be the *unilateral* Laplace transform—or simply *the* Laplace transform—of $f(q)$ and the latter the *inverse* Laplace transform of $f^L(s)$. The Parseval identity is

$$\int_0^\infty dq \exp(-2\gamma q) f(q)^* g(q) = (2\pi i)^{-1} \int_{\gamma - i\infty}^{\gamma + i\infty} ds f^L(s)^* g^L(s), \tag{8.9c}$$

where γ is larger than the types of $f(q)$ and $g(q)$.

8.1.5. On Bromwich Contours

A few remarks are in order as the following feature of (8.9) might appear puzzling: Having introduced an upper bound γ for the growth of $f(q)$ into the definition (8.7), we end up with an integration contour $\int_{\gamma - i\infty}^{\gamma + i\infty}$ which

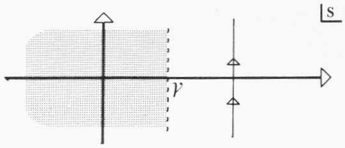


Fig. 8.4. Existence region (unshaded) for the unilateral Laplace transform and integration contour for the inverse transform.

depends on γ but of which we seem to have no clue before the integral is performed. Actually we do, as recalling one of the main results of Section 7.4 will show. Following Eq. (7.126) we proved that the Fourier transform of (8.7) is (a) an entire analytic function in the lower complex half-plane $\operatorname{Im} p < 0$ and (b) bounded by a constant C_f (as $a = 0$). As here $s = \gamma - \operatorname{Im} p + i \operatorname{Re} p$, *the function* $f^L(s)$ *will be entire analytic in the right half-plane* $\operatorname{Re} s > \gamma$. *The value of* γ *is thus a left bound for the region of analyticity of the function* $f^L(s)$. See Fig. 8.4. When we perform the inverse Laplace transform (8.9a), the integration path is such that $f^L(s)$ is analytic and bounded to its right and all singularities are confined to its left. Such integration paths are referred to as *Bromwich* contours. Clearly, for $q < 0$, the exponential factor $\exp(qs)$ in (8.9a) allows us to invoke Cauchy and Jordan and close the integration contour with a semicircle at infinity, obtaining $f(q) = 0$ for this region [Fig. 8.5(a)]. For $q > 0$ the integration requires more effort but can usually be dealt with by applying Cauchy and Jordan for poles and other techniques for branch cuts [Fig. 8.5(b)].

8.1.6. Example

Consider an example,

$$f_{n,c}(q) := \begin{cases} q^n \exp(cq), & q > 0, \\ 0, & q \leqslant 0, \end{cases} \tag{8.10}$$

which is of growth (1, Re c). The construction of its Laplace transform pro-

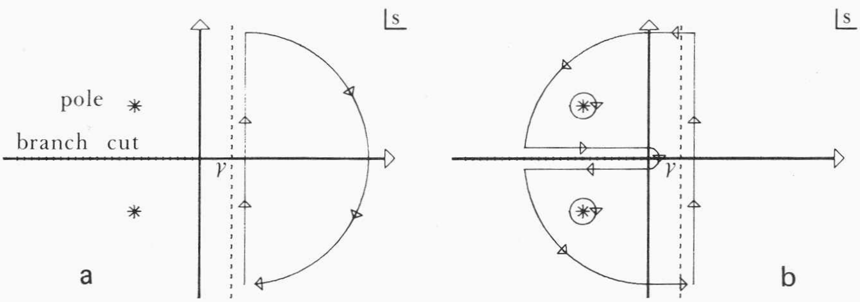


Fig. 8.5. Bromwich integration contours for the inverse unilateral transform for (a) $q < 0$ and (b) $q > 0$. A pair of conjugate poles and a branch cut have been assumed for the analytic continuation of the transform function.

ceeds by (8.7) with a choice of $\gamma > \operatorname{Re} c$, which makes the function integrable. Its Laplace transform can be found by successive integration by parts as

$$\begin{aligned} f^{L}_{n,c}(s) &= \int_0^\infty dq q^n \exp[-q(s-c)] \\ &= -(s-c)^{-1} q^n \exp[-q(s-c)]|_{q=0}^{\infty} \\ &\quad + n(s-c)^{-1} \int_0^\infty dq q^{n-1} \exp[-q(s-c)] \\ &= \cdots = n!\,(s-c)^{-n} \int_0^\infty dq \exp[-q(s-c)] = n!\,(s-c)^{-n-1}, \end{aligned}$$

$$\operatorname{Re}(s-c) = \gamma - \operatorname{Re} c > 0, \quad (8.11)$$

i.e., it is a function with an $(n+1)$-fold pole at $s = c$. The integration (8.11) is properly valid only for $\operatorname{Re} s > \operatorname{Re} c$, so the inverse Laplace transform along a vertical path at γ is inside this region, with $f^L(s)$ free of singularities to the right of it. The function $f^L(s)$, however, possesses an *analytic continuation* to the whole complex s-plane which allows its inverse transformation by means of the Cauchy and Jordan results. The former states that

$$(2\pi i)^{-1} \oint ds (s-c)^{-n-1} g(s) = (n!)^{-1} d^n g(s)/ds^n|_{s=c}, \quad (8.12)$$

while the latter tells us that for $q > 0$ we can set up a Bromwich contour with vanishing contribution at the infinite semicircle, so that $f(q)$ is regained as

$$\begin{aligned} (2\pi i)^{-1} \int_{\gamma - i\infty}^{\gamma + i\infty} & ds [n!\,(s-c)^{-n-1}] \exp(sq) \\ &= (2\pi i)^{-1} n! \oint ds (s-c)^{-n-1} \exp(sq) \\ &= \frac{d^n}{ds^n} [\exp(sq)]\bigg|_{s=c} = q^n \exp(cq). \quad (8.13) \end{aligned}$$

In Table 8.1 we have listed some useful Laplace transform pairs. Much more extensive tables can be found in the Bateman manuscript project (Erdelyi *et al.* (1954, Vol. I, Chapters IV and V) and in a recent table by Oberhettinger and Badii (1973).

In most applications it is the *inverse* Laplace transform of a function which yields the final solution to the problem. Thus it is the second part of the above example which should be of primary interest. It tells us that the inverse transform of a simple pole ($n = 0$) at $s = c$ is an exponential function $\exp(cq)$ times $(n!)^{-1}$. *Pairs* of poles at $c = a \pm ib$ will thus inverse-transform to oscillating functions *sin* or *cos*, depending on the relative residue signs.

Exercise 8.2. Verify the pairs of Laplace transforms of Table 8.1 where $f^L(s)$ is a function of the kind discussed above.

8.1.7. Derivatives and Boundary Conditions

The second main ingredient in the solution of differential equations by Laplace transformation is the way in which derivatives of functions transform and initial conditions appear. Assume $f(q)$ is differentiable as many times as required and that all its derivatives grow, for $q > 0$, not faster than $(1, \gamma)$ for some common γ. Then if $f^L(s)$ is the Laplace transform of $f(q)$, the transform of $f'(q) := df(q)/dq$ can be found from (8.9b) by integration by parts [in doing so we assume $f'(q)$ is continuous on $(0, \infty)$; see Exercise 8.4]:

$$\begin{aligned}(\mathbb{L}\mathbf{f}')(s) &= \int_0^\infty dq f'(q) \exp(-sq) \\ &= f(q)\exp(-sq)\big|_{q=0}^{\infty} + s\int_0^\infty dq f(q)\exp(-sq) \\ &= -f(0) + s(\mathbb{L}\mathbf{f})(s). \end{aligned} \tag{8.14}$$

The important thing about (8.14) is that $(\mathbb{L}\mathbf{f}')(s)$ is $s(\mathbb{L}\mathbf{f})(s)$—a result obtainable by Fourier transformation alone—*plus* the boundary value $f(0)$ of the transforming function. The second derivative is as easy to calculate and yields

$$(\mathbb{L}\mathbf{f}'')(s) = -f'(0) - sf(0) + s^2(\mathbb{L}\mathbf{f})(s). \tag{8.15}$$

The case for higher derivatives is included in Table 8.2: The boundary values and derivatives up to the order of differentiation minus one appear. For many differential equations this is all that is needed to determine the solution uniquely.

8.1.8. The Driven, Damped Oscillator

As an example where the boundary conditions appear, let us draw upon our old driven, damped harmonic oscillator system whose equation of motion is

$$\left(M\frac{d^2}{dq^2} + c\frac{d}{dq} + k\right)f(q) = F(q) \tag{8.16}$$

[see Eqs. (2.1) and (7.111), except that here we do not need to restrict c to positive values]. Using (8.14) and (8.15), we find the Laplace transform of (8.16) to be

$$M[-f'(0) - sf(0) + s^2 f^L(s)] + c[-f(0) + sf^L(s)] + kf^L(s) = F^L(s). \tag{8.17}$$

From here we can easily solve for $f^L(s)$:

$$\begin{aligned} f^L(s) &= (Ms^2 + cs + k)^{-1}\{F^L(s) + [Mf'(0) + (Ms + c)f(0)]\} \\ &:= f_F{}^L(s) + f_B{}^L(s). \end{aligned} \tag{8.18}$$

The structure of (8.18) is rather transparent: it contains a "stationary" term $f_F{}^L(s)$ due to the driving force transform $F^L(s)$ plus a transient response to the boundary conditions $f_B{}^L(s)$. The latter is immediately invertible: the two poles of the denominator are located by factoring the expression

$$Ms^2 + cs + k = M(s - s_+)(s - s_-). \tag{8.19a}$$

$$s_\pm := -\Gamma \pm ip_e, \qquad \Gamma := c/2M, \qquad p_e := (p_0{}^2 - \Gamma^2)^{1/2}, \qquad p_0 := (k/M)^{1/2}, \tag{8.19b}$$

where we have used the same variables as in (7.113). The inverse Laplace transform of $f_B^L(s)$ is zero for $q < 0$ since the integration path follows any abscissa $\gamma > \Gamma$. For $q > 0$ the integral can be found by closing the Bromwich contour around the denominator roots:

$$\begin{aligned} f_B(q) &= (2\pi i)^{-1} \int_{\gamma - i\infty}^{\gamma + i\infty} ds\,[(s - s_+)(s - s_-)]^{-1}[f'(0) + (s + c/M)f(0)] \\ &= +(s_- - s_+)^{-1}[f'(0) + (s_- + c/M)f(0)]\exp(s_- q) \\ &\quad + (s_+ - s_-)^{-1}[f'(0) + (s_+ + c/M)f(0)]\exp(s_+ q) \\ &= \exp(-\Gamma t)\{f(0)[\cos(p_e q) + \Gamma \sin(p_e q)/p_e] + f'(0)\sin(p_e q)/p_e\} \\ &= f(0)[cG(q) + M\dot{G}(q)] + Mf'(0)G(q), \end{aligned} \tag{8.20}$$

where

$$G(q) := \begin{cases} \exp(-\Gamma q)\sin(p_e q)/Mp_e, & q > 0, \\ 0, & q \leqslant 0, \end{cases} \tag{8.21}$$

is the Green's function of the system and $\dot{G}(q)$ its derivative. In obtaining this result we have used (8.13) with $n = 0$ and $s_\pm$ for c and collected terms. The transient response is identical to the corresponding results we have previously obtained from (2.10)–(2.13) and (7.115)–(7.122).

As to the stationary solution term $f_F{}^L(s)$ we see that it is $F^L(s)$ multiplied by the reciprocal of (8.19). Our intuition should tell us that the inverse Laplace transform of this product is a convolution—Laplace version—of the inverse transforms of the factors. In fact, it is exactly (7.117), namely,

$$f_F(q) = \int_0^q dq'\,F(q')G(q - q'), \tag{8.22}$$

with the understanding that the driving force $F(q)$ is, as are all functions, subject to unilateral Laplace transformation, zero up to $q = 0$.

Exercise 8.3. Starting from the relation between product and convolution under Fourier transformation, show that the unilateral Laplace version of this correspondence is as given in Table 8.2. Note that the abscissa of the integration path must be *larger* than the type of the factors.

Exercise 8.4. Show that if the function $f'(q)$ in (8.14) is discontinuous at some point d, its Laplace transform has an extra term involving the discontinuity of the function at that point, as shown in Table 8.2. Examine the case where there is more than one such point.

Exercise 8.5. Prove the rest of the entries in Table 8.2.

The Laplace transform has been used to solve Eq. (8.16) once more. In terms of directness and ease, the Laplace methods seems to be preferable to the Fourier transform, as the latter does not allow for growing functions without requiring Dirac δ's. The "cutting" process of Section 7.4 for negative q's also involves some effort. For these reasons, the unilateral Laplace transform has found wide acceptance as a tool in engineering and electronic computation. Texts centering on this method include (among many others) those of Gardner and Barnes (1942), Doetsch (1950, 1955, and 1961), and Craig (1964). Most books dealing with Fourier transforms will also have a chapter on Laplace transforms. The considerable mathematical interest of the latter has merited a few volumes by itself, such as the treatise by Widder (1941), Smith (1966), and Kuhfittig (1978). It is with some misgivings that we close this section having presented only the barest essentials of the subject. Function vector space concepts such as orthogonality and completeness of a basis, however, seem to be easier to develop in terms of Fourier—and similar *unitary*—transforms.

Table 8.1 Some Useful Laplace Transform Pairs

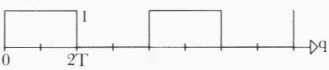
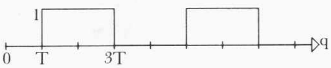
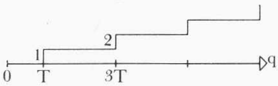
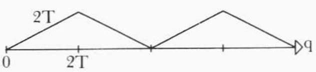
$f(q)$	$f^L(s)$	Domain
$\exp(cq)$	$(s - c)^{-1}$	Re $s > c$
$\sin cq$	$c/(s^2 + c^2)$	Re $s > 0$
$\cos cq$	$s/(s^2 + c^2)$	Re $s > 0$
$\Theta(q - a), a > 0$	$s^{-1} \exp(-as)$	Re $s > 0$
	$\{s[1 + \exp(-2Ts)]\}^{-1}$	Re $s > 0$
	$(s \cosh Ts)^{-1}$	Re $s > 0$
	$(s \sinh Ts)^{-1}$	Re $s > 0$
	$s^{-2} \tanh Ts$	Re $s > 0$
$q^n \exp(cq)$	$n!\,(s - c)^{-n-1}$	Re $s > c$
$\exp(-q^2/2a^2)$	$a(\pi/2)^{1/2} \exp(a^2 s^2/2)\, \mathrm{erfc}(2^{-1/2} as)$	All $s \in \mathscr{C}$
$J_\mu(cq), \mu > -1$	$(s^2 + c^2)^{-1/2} c^{-\mu}[(s^2 + a^2)^{1/2} - s]^\mu$	Re $s > 0$

Table 8.2 Laplace Transform under Various Operators and Operations

Relation	$f(q)$	$f^L(s)$
Linear combination	$af(q) + bg(q)$	$af^L(s) + bg^L(s)$
Translation	$f(q - a)$	$\exp(-as)\left[f^L(s) + \int_{-a}^{0} dq' f(q') \exp(-sq')\right]$
	$\exp(aq)f(q)$	$f^L(s - a)$
Dilatation	$f(q/c)$, $c > 0$	$cf^L(cs)$
Multiplication	$f(q)g(q)$	$(2\pi i)^{-1} \int_{\gamma - i\infty}^{\gamma + i\infty} ds' f^L(s')g^L(s - s')$, $\gamma >$ type f, g
Convolution	$\int_0^q dq' f(q')g(q - q')$	$f^L(s)g^L(s)$
Differentiation	$\frac{d^n}{dq^n} f(q)$, continuous	$s^n f^L(s) - \sum_{m=1}^{n} s^{m-1} f^{(n-m)}(0)$
	$\frac{d}{dq} f(q)$, discontinuous at $q = d$	$sf^L(s) - f(0) + \exp(-dq)[f(d^-) - f(d^+)]$
	$(-q)^n f(q)$	$\frac{d^n}{ds^n} f^L(s)$
Integration	$\int_0^q dq' f(q')$	$s^{-1} f^L(s)$
	$-q^{-1} f(q)$	$\int_0^s ds' f(s')$

8.2. Mellin Transforms

Mellin transforms are closely related to Fourier transforms and constitute a "continuous analogue" of Taylor series. As was the case with Laplace transforms, there are at least two versions of this transform, a bilateral and a unilateral one. The first will help us to establish Dirac orthonormality and completeness relations for the repulsive oscillator wave functions. The second is useful for several of its properties involving convolution transformation of differential equations into difference relations and the appearance of gamma functions.

8.2.1. Positive, Negative, and Bilateral Mellin Transforms

Consider the direct and inverse Fourier transform equations (7.1) for functions $g(q)$ and $\tilde{g}(p)$ and make the change of variables $q =: \ln x$, so that $\exp(\pm ipq) = x^{\pm ip}$, mapping the real line q onto the positive half-line x. As $dq = dx/x$, it is convenient to attach a factor $x^{-1/2}$ to the kernel x^{ip} and a factor $x^{-1/2}$ to $g(\ln x)$, denoting the new function by $f(x)$. After this has been done, the Fourier transform pair is

$$f(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} d\lambda f_+{}^M(\lambda) x^{i\lambda - 1/2}, \qquad x \in (0, \infty), \tag{8.23a}$$

$$f_+{}^M(\lambda) = (2\pi)^{-1/2} \int_0^{\infty} dx f(x) x^{-i\lambda - 1/2}, \qquad \lambda \in \mathscr{R}, \tag{8.23b}$$

where, in addition, we have changed the dummy variable p by λ and $\tilde{g}(p)$ by $f_+{}^M(\lambda)$. The function $f_+{}^M(\lambda)$ will be called the *positive Mellin transform* of $f(x)$. Effecting the same changes of variable and notation, the Parseval identity (7.1c) becomes

$$\int_0^{\infty} dx f(x)^* g(x) = \int_{-\infty}^{\infty} d\lambda f_+{}^M(\lambda)^* g_+{}^M(\lambda). \tag{8.23c}$$

The transform pair (8.23a)–(8.23b) is somewhat lopsided, since the function $f(x)$ is allowed to have only a positive argument. Negative values of x can be admitted only if, to begin with, we had introduced a change of variables $q =: \ln(-x)$, $x < 0$. By following through with the same substitutions, this leads to

$$f(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} d\lambda f_-{}^M(\lambda)(-x)^{i\lambda - 1/2}, \qquad x \in (-\infty, 0), \tag{8.24a}$$

$$f_-{}^M(\lambda) = (2\pi)^{-1/2} \int_{-\infty}^{0} dx f(x)(-x)^{-i\lambda - 1/2}, \qquad \lambda \in \mathscr{R}, \tag{8.24b}$$

and the Parseval identity

$$\int_{-\infty}^{0} dx f(x)^* g(x) = \int_{-\infty}^{\infty} d\lambda f_-{}^M(\lambda)^* g_-{}^M(\lambda). \tag{8.24c}$$

Correspondingly, $f_-{}^M(\lambda)$ will be called the *negative Mellin transform* of $f(x)$. As, clearly, the positive and negative halves of $f(x)$ are in general unrelated, the two transforms $f_+{}^M(\lambda)$ and $f_-{}^M(\lambda)$ are independent. If we introduce the positive—and negative—x-function (7.202b),

$$x_+ := \begin{cases} x, & x > 0, \\ 0, & x \leqslant 0, \end{cases} \qquad x_- := \begin{cases} 0, & x \geqslant 0, \\ -x, & x < 0, \end{cases} \tag{8.25}$$

we can *join* (8.23) and (8.24) for $x \in \mathscr{R}$ as

$$(\mathbb{M}_B^{-1} \mathbf{f}^{BM})(x) := f(x) = (2\pi)^{-1/2} \sum_{\sigma = \pm} \int_{-\infty}^{\infty} d\lambda f_\sigma^{BM}(\lambda) x_\sigma^{i\lambda - 1/2}, \tag{8.26a}$$

$$(\mathbb{M}_B \mathbf{f})_\sigma(\lambda) := f_\sigma^{BM}(\lambda) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} dx f(x) x_\sigma^{-i\lambda - 1/2}, \tag{8.26b}$$

$$\int_{-\infty}^{\infty} dx f(x)^* g(x) = \sum_{\sigma = \pm} \int_{-\infty}^{\infty} d\lambda f_\sigma^{BM}(\lambda)^* g_\sigma^{BM}(\lambda). \tag{8.26c}$$

The *pair* of functions $\{f_\sigma^{BM}(\lambda)\}_{\sigma=\pm}$, $\lambda \in \mathscr{R}$, are called the *bilateral* Mellin transform of $f(x)$, $x \in \mathscr{R}$. We must stress that the two component functions $f_\pm^{BM}(\lambda)$ are needed to reconstitute $f(x)$ for $x \in \mathscr{R}$. The latter is the inverse bilateral Mellin transform of the pair $f_\pm^{BM}(\lambda)$.

8.2.2. Orthogonality and Completeness of $x_\pm^{i\lambda-1/2}$

The bilateral Mellin synthesis (8.26a) can be seen as the continuous analogue of the Taylor expansion. Whereas the latter sums over the positive integer powers of x, the former involves integration of powers along a line in the complex plane. This is represented schematically in Fig. 8.6. Actually, pairs of series and transforms occur in several other instances, as will be mentioned in Section 8.5. Last, as we have only performed a change of variable and function in passing from the Fourier transform to the bilateral Mellin transform (8.26), the powerful results of the former can be translated to the latter *verbatim*.

One of the results of Section 7.3 was to justify that if one of the functions of an integral transform pair was introduced in the other and the integrals exchanged, a representation of the Dirac δ by a divergent integral was obtained. Following this procedure for the bilateral Mellin transform, substituting (8.26a) into (8.26b), we obtain the *orthogonality* relation for the set of functions $\{(2\pi)^{-1/2}x_\sigma^{-i\lambda-1/2}\}_{\sigma=\pm,\lambda\in\mathscr{R}}$ as

$$(2\pi)^{-1}\int_{-\infty}^{\infty} dx\, x_\sigma^{i\lambda-1/2}x_{\sigma'}^{-i\lambda'-1/2} = \delta(\lambda-\lambda')\delta_{\sigma,\sigma'}, \tag{8.27}$$

where $\delta_{\sigma,\sigma'}$ is the ordinary Kronecker δ in the indices σ and σ' and $\delta(\lambda-\lambda')$ the Dirac δ in the index λ. Similarly, by substituting (8.26b) into (8.26a) and exchanging integrals, the *completeness* relation

$$(2\pi)^{-1}\sum_{\sigma=\pm}\int_{-\infty}^{\infty} d\lambda\, x_\sigma^{i\lambda-1/2}x_\sigma'^{-i\lambda-1/2} = \delta(x-x') \tag{8.28}$$

is obtained. The set of functions $\{(2\pi)^{-1/2}x_\sigma^{i\lambda-1/2}\}_{\sigma=\pm,\lambda\in\mathscr{R}}$ thus constitutes a generalized (Dirac) orthonormal basis for $\mathscr{L}^2(\mathscr{R})$. Equations (8.27)–(8.28) are valid for the positive or negative Mellin transforms separately if we disregard the index σ and restrict integration and δ's to positive or negative values of x.

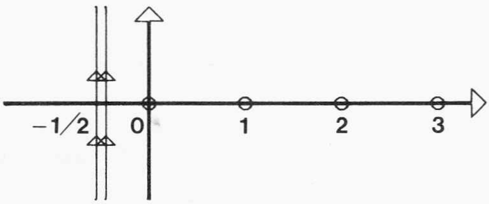


Fig. 8.6. The bilateral Mellin transform (double integration contour at $-\frac{1}{2}+i\lambda$) as a continuous analogue of the Taylor series (circles on the integer points).

Exercise 8.6. Derive the orthogonality relation (8.27) from (7.93) for $n = 0$, $q = \lambda - \lambda'$, and $p = \ln(\pm x)$ for the supports of $x_{\pm}$. Similarly, derive the completeness relation (8.28) from (7.93) for $n = 0$, $p = \lambda$, $q = \ln(\pm x)$, $q' = \ln(\pm x')$ in the appropriate ranges. You will be faced with a δ in $\ln x - \ln x'$ for which (7.96) can be used.

8.2.3. Completeness of the Repulsive Oscillator Wave Functions

The results (8.27) and (8.28) lead us neatly to the orthogonality and completeness relation for the repulsive oscillator wave functions presented in Section 7.5. The $\tilde{v}_\lambda{}^{\pm}(p)$ functions in Eqs. (7.202), which were instrumental in the solution of the problem, are (for $\sigma = 1$) $p_{\pm}^{-i\lambda-1/2}$ times the phase $\exp(ip^2/4)$, which is independent of λ. Now multiplication of the set in (8.27)–(8.28) by a purely x-dependent phase leaves the *inner product* (8.27) invariant: the phase of the first function cancels the phase of the second. Similarly, when multiplied by $\exp(ix^2/4)\exp(-ix'^2/4)$ on both sides the λ-integral (8.28) yields completeness for the $\tilde{v}_\lambda{}^{\pm}(p)$, as the left-hand side is nonzero only for $x = x'$. Since the Fourier transform is unitary—and for the full explanation of *this* fact we have to rely on more general results—it will map a Dirac basis of $\mathscr{L}^2(\mathscr{R})$ onto another such basis. Thus the set of functions $v_\lambda{}^{\pm}(q)$ constitutes a Dirac basis of $\mathscr{L}^2(\mathscr{R})$ as well. Finally, multiplication by the λ-independent phase factor $\exp(iq^2/2)$ and the q-independent one $2^{i\lambda/2}$ validates the set $\{\chi_\lambda{}^{\sigma}(q)\}_{\sigma=\pm,\lambda\in\mathscr{R}}$ in (7.203) as a Dirac generalized basis, orthogonal and complete in $\mathscr{L}^2(\mathscr{R})$.

8.2.4. Unilateral Mellin Transforms

As with Laplace transforms, there are at least two versions of the Mellin transform, the bilateral one sketched above and the more usual *Mellin–Laplace*, or simply *the* Mellin, transform. We shall now detail the construction of the latter and mention some of its properties and areas of application. We start again from the Fourier transform pair (7.1) for $g(q)$ and $\tilde{g}(p)$, assuming that for some nonempty range of γ, $\exp(\gamma q)g(q)$ is integrable (this may be true only for $\gamma = 0$). We set $r := \gamma + ip$ and $u := \exp(-q) > 0$, following through with the changes in differentials and integration ranges. Finally, we introduce the functions $f(u) := (2\pi)^{-1/2}\exp(\gamma q)g(q)$ and $f^M(r) := \tilde{g}(p)$, obtaining for them the relation

$$(\mathbb{M}^{-1}\mathbf{f}^M)(u) := f(u) = (2\pi i)^{-1}\int_{\gamma-i\infty}^{\gamma+i\infty} dr f^M(r)u^{-r}, \tag{8.29a}$$

$$(\mathbb{M}\mathbf{f})(r) := f^M(r) = \int_0^\infty du f(u)u^{r-1} \tag{8.29b}$$

and the Parseval formula

$$\int_0^\infty du f(u)^* g(u)u^{2\gamma-1} = (2\pi i)^{-1}\int_{\gamma-i\infty}^{\gamma+i\infty} dr f^M(r)^* g^M(r). \tag{8.29c}$$

8.2.5. Example

Working out an example will clarify some points. Consider

$$F_{n,-c}(u) := u^n \exp(-cu), \qquad c \in \mathscr{C}, \operatorname{Re} c > 0, u > 0, \tag{8.30a}$$

which represents an exponentially damped oscillating function. As the integral (8.29b) over $(0, \infty)$ is convergent for $\operatorname{Re}(r + n) > 0$, the Mellin transform over this region is

$$\begin{aligned} F^M_{n,-c}(r) &= \int_0^\infty du \exp(-cu) u^{r+n-1} \\ &= c^{-r-n} \int_0^\infty du' \exp(-u') u'^{r+n-1} = c^{-r-n}\Gamma(r + n), \\ &\qquad \operatorname{Re}(r + n) > 0. \end{aligned} \tag{8.30b}$$

In changing variables for $\operatorname{Im} c \neq 0$ the integration contour is made along a ray in the complex u'-plane and shifted back as in Fig. 7.13 by the use of the Cauchy and Jordan results. The remaining integral is the gamma function (see Appendix A) for $\operatorname{Re}(r + n) > 0$, and this by analytic continuation defines $F^M_{n,c}(r)$ for all complex $r + n \neq 0, -1, -2, \ldots$. One of the useful characteristics of the Mellin transform is that, as we saw, exponential functions are transformed into gamma functions, whose difference relations [i.e., those relating $\Gamma(z)$ with $\Gamma(z \pm n)$] are well known. Compare this with the Laplace transform of the same example, Eq. (8.11), which is a function with an $(n + 1)$th-order pole at c. The original function in the Fourier transform pair giving rise to (8.30a) is $\sim\exp[-(\gamma + n)q - c\exp(-q)]$ for any $\gamma > -n$ which is integrable on $q \in \mathscr{R}$. The Bromwich contour yielding the inverse Mellin transform of (8.30b) is thus along a vertical path in the complex r-plane crossing the real axis at γ, to the right of all the poles of the function, as in Fig. 8.7. The inverse transform of (8.30b) can thus be found, for $u > 0$, as

$$\begin{aligned} &(2\pi i)^{-1} \int_{\gamma - i\infty}^{\gamma + i\infty} dr\,\Gamma(r + n)(uc)^{-r} c^{-n} \\ &\quad = (2\pi i)^{-1} c^{-n} \sum_{m=-n}^{-\infty} \oint_{Cm} dr\,\Gamma(r + n)(uc)^{-r} \\ &\quad = c^{-n} \sum_{m=-n}^{-\infty} [\operatorname{Res} \Gamma(r + n)|_{r=m}](uc)^{-m} \\ &\quad = c^{-n} \sum_{k=0}^{\infty} (-1)^k (uc)^{n+k}/k! = u^n \exp(-cu) = F_{n,-c}(u). \end{aligned} \tag{8.30c}$$

In the process, we have used the Cauchy–Jordan results to reduce the Bromwich contour to a series of contours C_m enclosing the integrand poles a $-n, -n - 1, \ldots$ and the residue formula for the gamma function at these

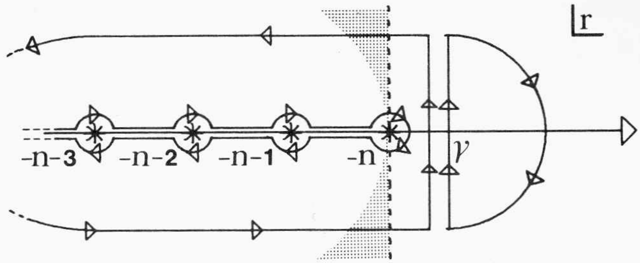


Fig. 8.7. Integration contour for Eq. (8.30c).

points [see Eq. (A.7)]. The exponential series is thus regained, and the correct original function follows. Several functions and their Mellin transforms are listed in Table 8.3. Extensive tables of Mellin transforms can be found in the Bateman manuscript project [see Erdelyi *et al.* (1954, Chapters VI and VII)].

Exercise 8.7. Check that for $u > 0$ the Bromwich contour cannot be closed through a right semicircle. Explore the situation for complex u.

The convergence requirements of the Mellin integral (8.29b) may fail to be met by many functions of interest. The inconvenient growth of a function $g(q)$ for $q \to -\infty$ $(u \to \infty)$ was cured by the introduction of a factor $\exp(\gamma q)$. A similar procedure could solve growth problems for $q \to +\infty$ $(u \to 0)$ with a factor $\exp(\gamma' q)$. The simultaneous correction of divergences at both ends, however, may be impossible. The Mellin transform (8.29b) can then be broken up into two Mellin transforms of functions with support on $(0, 1)$ and $[1, \infty)$ and appropriate half-planes γ and γ'. Such a procedure is followed in Morse and Feshbach (1953, p. 976). For the following results, we shall simply assume that a nonvanishing common band of convergence in the complex r-plane exists for all functions involved.

8.2.6. Further Properties

As with Fourier transforms, the properties of differentiation, multiplication by a power of the argument, and translation under Mellin transformation point toward the possible applications of this transform. Consider the Mellin transform of the derivative of a function $f'(u) := df(u)/du$ and its subsequent integration by parts:

$$(\mathbb{M}\mathbf{f}')(r) = \int_0^\infty du[df(u)/du]u^{r-1}$$

$$= f(u)u^{r-1}|_0^\infty - (r-1)\int_0^\infty du f(u)u^{r-2}. \tag{8.31}$$

If, now, within a band in the r-plane the integrand vanishes at the interval ends, the constant term in (8.31) will be zero, and Eq. (8.31) will equal

$-(r-1)(\mathbb{M}\mathbf{f})(r-1)$. This can easily be generalized for the nth derivative using the operator $\mathbb{V}^n = d^n/du^n$:

$$(\mathbb{M}\mathbb{V}^n\mathbf{f})(r) = -\sum_{p=1}^{n} (-1)^p(r-1)(r-2)\cdots(r-p)f^{(n-p)}(u)u^{r-p}\Big|_{u=0}^{\infty} + (-1)^n(r-1)(r-2)\cdots(r-n)(\mathbb{M}\mathbf{f})(r-n). \tag{8.32a}$$

If all boundary values are zero, this reduces to

$$(\mathbb{M}\mathbb{V}^n\mathbf{f})(r) = (-1)^n[\Gamma(r)/\Gamma(r-n)](\mathbb{M}\mathbf{f})(r-n). \tag{8.32b}$$

We see that differentiation of the original function becomes essentially a *translation* of the transform's argument. Using integration by parts, Eqs. (8.32) can be validated for antiderivatives as well, for negative values of n. The integration constants must be set to zero. Pure translation of the transform's argument [i.e., without multiplication by $(r-p)$-factors] can be achieved by multiplying $f(u)$ by u^n. Using the multiplication by argument operator $\mathbb{Q}$ introduced in (7.55), we can write

$$(\mathbb{M}\mathbb{Q}^n\mathbf{f})(r) = (\mathbb{M}\mathbf{f})(r+n). \tag{8.33}$$

Equations (8.32) and (8.33) are of course valid for real or complex n within the convergence band of the Mellin integral for the function in question. They can be combined as in, say,

$$(\mathbb{M}\mathbb{Q}^m\mathbb{V}^n\mathbf{f})(r) = (\mathbb{M}\mathbb{V}^n\mathbf{f})(r+m) = (-1)^n[\Gamma(r+m)/\Gamma(r+m-n)](\mathbb{M}\mathbf{f})(r+m-n), \tag{8.34a}$$

$$(\mathbb{M}\mathbb{V}^n\mathbb{Q}^m\mathbf{f})(r) = (-1)^n[\Gamma(r)/\Gamma(r-n)](\mathbb{M}\mathbb{Q}^m\mathbf{f})(r-n) = (-1)^n[\Gamma(r)/\Gamma(r-n)](\mathbb{M}\mathbf{f})(r+m-n). \tag{8.34b}$$

Further properties of operator and operations under Mellin transformation can be found in Table 8.3.

Exercise 8.8. Note that as (8.34), for $m = 1 = n$, is valid for an arbitrary function with appropriate growth conditions, one can deduce from here that $[\mathbb{V}, \mathbb{Q}] := \mathbb{V}\mathbb{Q} - \mathbb{Q}\mathbb{V} = \mathbb{1}$ on any function in this space. Similarly, verify by algebraic manipulations on multinomials that (7.67) holds.

The peculiar relationship under Mellin transformation among differen-iation, multiplication by powers of the argument, and translations should not be surprising as that is precisely the behavior of x^r as a function of r under these operations.

8.2.7. Applications and References

One of the areas of application of the Mellin transform concerns the solution of the Laplace equation $\nabla^2 f(\mathbf{u}) = 0$ in two or more dimensions with certain boundary conditions. When ∇ is written in polar coordinates (u, ϕ) [see Eq. (6.16)] and is multiplied by u^2, it is

$$(u^2\, \partial^2/\partial u^2 + u\, \partial/\partial u + \partial^2/\partial\phi^2) f(u, \phi) = 0. \tag{8.35}$$

Mellin transformation of (8.35) with respect to the radial variable u leads, by (8.34a), to

$$(r^2 + \partial^2/\partial\phi^2) f^M(r, \phi) = 0, \tag{8.36}$$

whose solutions are of the form

$$f^M(r, \phi) = a(r) \cos r\phi + b(r) \sin r\phi. \tag{8.37}$$

The boundary conditions one can impose on $f(u, \phi)$ in order to fix $a(r)$ and $b(r)$ are, for instance, $f(u, \phi)$ as a function of u for two given values of ϕ, say ϕ_a and ϕ_b. These can represent, for instance, the electrostatic potential between two fixed, charged, nonconducting plates forming a wedge between ϕ_a and ϕ_b, the stress or the stationary temperature distribution between two such walls with fixed temperature. This problem, with a variety of boundary conditions, has been solved by Tranter (1948) and Lemon (1962). Essentially, the Mellin transforms of the boundary conditions are equated to (8.37) for $\phi = \phi_a$ and ϕ_b, respectively. The ensuing solutions are examined, for instance, in the books by Colombo (1959) and Sneddon (1972, Chapter 4).

Exercise 8.9. Examine more closely the conditions under which the Mellin transformation from (8.35) to (8.36) holds. Assume $f(u, \phi)$ behaves like u^{γ_1} for $u \to 0$ and u^{γ_2} for $u \to \infty$. Show that (8.36) holds for $-\gamma_1 < u < -\gamma_2$.

Exercise 8.10. Note that once the functions $a(r)$ and $b(r)$ in (8.37) have been found, the function (8.37) still has to be subject to an inverse Mellin transform. As the two functions in (8.37) contain the boundary data, one needs to know the inverse transform of a *product* of two functions. Prove the *convolution* formulas in Table 8.3.

Table 8.3 Mellin Transform under Various Operators and Operations

Relation	$f(u)$	$f^M(r)$
Linear combination	$af(u) + bg(u)$	$af^M(r) + bg^M(r)$
Translation	$u^n f(u)$	$f^M(r + n)$
	$d^m f(u)/du^m$	$(-1)^m[\Gamma(r)/\Gamma(r - m)] f^m(r - m)$
Differentiation	$(\ln u)^k f(u)$	$d^k f^M(r)/dr^k$
Dilatation	$f(au),\ a > 0$	$a^r f^M(r)$
	$b^{-1} f(u^{1/b})$	$f^M(br)$
Multiplication	$f(u)g(u)$	$(2\pi i)^{-1} \int_{\gamma - i\infty}^{\gamma + i\infty} dr' f^M(r') g^M(r - r')$
Convolution	$\int_0^\infty u'^{-1}\, du' f(u/u') g(u')$	$f^M(r) g^M(r)$

Table 8.4 Some Useful Mellin Transform Pairs

$f(u)$	$f^M(r)$	Domain
$u^n \exp(-cu)$, $\operatorname{Re} c > 0$	$c^{-r-n}\Gamma(r+n)$	$\operatorname{Re}(r+n) > 0$
$(1+u^n)^{-m}$	$\Gamma(r/n)\Gamma(m - r/n)/n\Gamma(m)$	$0 < \operatorname{Re} r < mn$
$\exp(-u^2/2\omega)$	$(2\omega)^{r/2}\frac{1}{2}\Gamma(r/2)$	$0 < \operatorname{Re} r$
$\cos au$	$a^{-r}\Gamma(r)\cos(\pi r/2)$	$0 < \operatorname{Re} r < 1$
$\sin au$	$a^{-r}\Gamma(r)\sin(\pi r/2)$	$0 < \operatorname{Re} r < 1$
$\dfrac{1 + u\cos\phi}{1 - 2u\cos\phi + u^2}$, $\lvert\phi\rvert < \pi$	$\pi \cos r\phi/\sin r\pi$	$0 < \operatorname{Re} r < 1$
$\dfrac{u\sin\phi}{1 - 2u\cos\phi + u^2}$, $\lvert\phi\rvert < \pi$	$\pi \sin r\phi/\sin r\pi$	$0 < \operatorname{Re} r < 1$
$u^{-\nu}J_\nu(au)$	$\dfrac{(a/2)^{\nu-r}\Gamma(r/2)}{2\Gamma(\nu - r/2 + 1)}$	$0 < \operatorname{Re} r < 1$

8.3. *N*-Dimensional Fourier Transforms

A straightforward generalization of the results for the Fourier transformation of functions of one variable is the consideration of functions of N variables and their corresponding N-fold Fourier transformation. Most results from the one-dimensional case can be "vectorized" by inspection.

8.3.1. Extension from One to *N* Dimensions

Consider a function $f(\mathbf{q})$ of the vector variable $\mathbf{q} = (q_1, q_2, \ldots, q_N)$. As a function of q_1 we can apply the Fourier transformation (7.1b)—assuming all necessary conditions are satisfied—and obtain a function $\tilde{f}^{[1]}(p_1, q_2, \ldots, q_N)$. This function in turn can be subject to the same transformation with respect to the variable q_2 and so on, obtaining finally

$$(\mathbb{F}_{(N)}^{-1}\tilde{\mathbf{f}})(\mathbf{q}) := f(\mathbf{q}) = (2\pi)^{-N/2} \int_{\mathscr{R}^N} d^N\mathbf{p}\tilde{f}(\mathbf{p}) \exp(i\mathbf{p}\cdot\mathbf{q}), \tag{8.38a}$$

$$(\mathbb{F}_{(N)}\mathbf{f})(\mathbf{p}) := \tilde{f}(\mathbf{p}) = (2\pi)^{-N/2} \int_{\mathscr{R}^N} d^N\mathbf{q} f(\mathbf{q}) \exp(-i\mathbf{p}\cdot\mathbf{q}), \tag{8.38b}$$

where

$$\int_{\mathscr{R}^N} d^N\mathbf{q} \cdots := \int_{-\infty}^{\infty} dq_1 \int_{-\infty}^{\infty} dq_2 \cdots \int_{-\infty}^{\infty} dq_N \cdots, \tag{8.38c}$$

and similarly for integration over $\mathbf{p}$-space. We have also used the familiar inner (or *scalar*) product notation $\mathbf{p}\cdot\mathbf{q} := p_1q_1 + p_2q_2 + \cdots + p_Nq_N$ in order to avoid confusion with the earlier sesquilinear product $(\mathbf{p}, \mathbf{q})$ in Part I. If $\mathbf{p}$ and $\mathbf{q}$ are represented as column vectors, $\mathbf{p}\cdot\mathbf{q} = \mathbf{p}^T\mathbf{q}$, where $\mathbf{x}^T$ is the transpose of vector (or matrix) $\mathbf{x}$. The Parseval identity is

$$(\mathbf{f}, \mathbf{g})_N := \int_{\mathscr{R}^N} d^N\mathbf{q} f(\mathbf{q})^* g(\mathbf{q}) = \int_{\mathscr{R}^N} d^N\mathbf{p} \tilde{f}(\mathbf{p})^* \tilde{g}(\mathbf{p}) = (\tilde{\mathbf{f}}, \tilde{\mathbf{g}})_N. \quad (8.38\text{d})$$

8.3.2. Linear Transformations of the Underlying Space

The properties of the N-dimensional Fourier transform under linear combination, convolution, translation, and differentiation are perfectly parallel to those of the one-dimensional transform in Chapter 7, except for some factors or exponents involving the value of N which are easy to ascertain. In Table 8.5 we have collected these results. Most can be found by inspection, "vectorizing" the corresponding one-dimensional expressions: replacing q, p, and qp by $\mathbf{q}$, $\mathbf{p}$, and $\mathbf{q}\cdot\mathbf{p}$; $\int_{-\infty}^{\infty} dq$ by $\int_{\mathscr{R}^N} d^N\mathbf{q}$; factors of $(2\pi)^{-1/2}$ by $(2\pi)^{-N/2}$; etc. For dilatations, however, we have a nontrivial generalization: general linear transformations in $\mathbf{q}$-space and corresponding ones in $\mathbf{p}$-space. To obtain them, assume $f(\mathbf{q})$ and its Fourier transform $\tilde{f}(\mathbf{p})$ are known. We wish to find, in terms of these, the transform of

$$(\mathbb{D}_\mathbf{A}\mathbf{f})(\mathbf{q}) = |\det \mathbf{A}|^{-1/2} f(\mathbf{A}^{-1}\mathbf{q}), \quad (8.39)$$

where $\mathbb{D}_\mathbf{A}$ is an operator which carries the action of the $N \times N$ matrix $\mathbf{A}$, which we assume to be real and nonsingular ($\det \mathbf{A} \neq 0$). Equation (8.39) is the natural generalization of Eq. (7.34). A change of variable $\mathbf{q}' := \mathbf{A}^{-1}\mathbf{q}$ yields

$$\begin{aligned}(\mathbb{F}_{(N)}\mathbb{D}_\mathbf{A}\mathbf{f})(\mathbf{p}) &= |\det \mathbf{A}|^{-1/2}(2\pi)^{-N/2} \int_{\mathscr{R}^N} d^N\mathbf{q} f(\mathbf{A}^{-1}\mathbf{q}) \exp(-i\mathbf{p}\cdot\mathbf{q}) \\ &= |\det \mathbf{A}|^{1/2}(2\pi)^{-N/2} \int_{\mathscr{R}^N} d^N\mathbf{q}' f(\mathbf{q}') \exp(-i\mathbf{p}\cdot\mathbf{A}\mathbf{q}') \\ &= |\det \mathbf{A}|^{1/2}(\mathbb{F}\mathbf{f})(\mathbf{A}^T\mathbf{p}) = (\mathbb{D}_{\mathbf{A}^{T-1}}\mathbb{F}_{(N)}\mathbf{f})(\mathbf{p}) \quad (8.40)\end{aligned}$$

since $\mathbf{p}^T\mathbf{A}\mathbf{q}' = (\mathbf{A}^T\mathbf{p})^T\mathbf{q}'$ and $d^N\mathbf{q} = \det \mathbf{A}\, d^N\mathbf{q}'$ is the transformation Jacobian. When $\det \mathbf{A} < 0$, i.e., as in a reflection through the origin of an odd number of coordinate axes, an odd number of integrations will have the usual bound order inverted. A reversal of these integration limits will cancel the sign of $\det \mathbf{A}$ and yield an absolute valued factor $|\det \mathbf{A}|$.

Exercise 8.11. Show that the dilatation operator $\mathbb{D}_\mathbf{A}$ is *unitary*, i.e., $(\mathbb{D}_\mathbf{A}\mathbf{f}, \mathbb{D}_\mathbf{A}\mathbf{g})_N = (\mathbf{f}, \mathbf{g})_N$ for all $\mathbf{f}$ and $\mathbf{g}$ for which the inner product is finite. This parallels Exercise 7.9–10.

Exercise 8.12. Verify that $\mathbb{D}_{\mathbf{A}}\mathbb{D}_{\mathbf{B}} = \mathbb{D}_{\mathbf{AB}}$ applied to any function $f(\mathbf{q})$. This generalizes Exercise 7.9.

Exercise 8.13. Find the N-dimensional convolution forms of Table 8.5.

Exercise 8.14. Define the multiplication-by-q_j operator as $\mathbb{Q}_j$ and the differentiation operator $\mathbb{P}_k := -i\partial/\partial q_k$ as generalizations of (7.55) and (7.56). Show that

$$\mathbb{F}_{(N)}\mathbb{P}_j\mathbb{F}_{(N)}^{-1} = \mathbb{Q}_j, \qquad \mathbb{F}_{(N)}\mathbb{Q}_k\mathbb{F}_{(N)}^{-1} = -\mathbb{P}_k. \tag{8.41}$$

Clearly, also

$$[\mathbb{Q}_j, \mathbb{P}_k] := \mathbb{Q}_j\mathbb{P}_k - \mathbb{P}_k\mathbb{Q}_j = i\mathbb{1}, \tag{8.42}$$

as in (7.59).

Exercise 8.15. Define the N-dimensional *dispersion* of a function $f(\mathbf{q})$ as

$$\Delta_f^{(N)} := \left[\int_{\mathscr{R}^N} d^N\mathbf{q}\,|f(\mathbf{q})|^2(\mathbf{q} - \overline{\mathbf{q}^1})^2\right] \Big/ \left[\int_{\mathscr{R}^N} d^N\mathbf{q}\,|f(\mathbf{q})|^2\right], \tag{8.43}$$

where $\overline{\mathbf{q}^1}$ is the vector *average* (or first moment) of $f(\mathbf{q})$, the analogue of (7.216) for $r = 1$ and vector $\mathbf{q}$. From (8.41) and (8.42), show that the uncertainty relation (7.218) becomes

$$\Delta_f^{(N)}\Delta_{\tilde{f}}^{(N)} \geqslant N/4. \tag{8.44}$$

What happens with the equivalent width relation (7.223)?

The relation between linear transformations in $\mathbf{q}$- and $\mathbf{p}$-spaces embodied in Eq. (8.40) has a very important particular case: if the transformation matrix $\mathbf{A}$ is an *orthogonal* matrix [i.e., an angle-preserving transformation so that

$$(\mathbf{A}\mathbf{q}_1)\cdot\mathbf{A}\mathbf{q}_2 = \mathbf{q}_1{}^T\mathbf{A}^T\mathbf{A}\mathbf{q}_2 = \mathbf{q}_1{}^T\mathbf{q}_2 = \mathbf{q}_1\cdot\mathbf{q}_2$$

for every $\mathbf{q}_1, \mathbf{q}_2 \in \mathscr{R}^N$], then $\mathbf{A}^T = \mathbf{A}^{-1}$, $\det \mathbf{A} = \pm 1$, and the transformations in $\mathbf{q}$- and $\mathbf{p}$-space are the *same*. In terms of operators this means that, for $\mathbf{A}$ orthogonal, $\mathbb{D}_{\mathbf{A}}$ and $\mathbb{F}_{(N)}$ *commute*. In the $N = 1$ case, the analogue of an orthogonal matrix is multiplication by ± 1, and in Section 7.2 we saw that parity was preserved under Fourier transformations. For $N > 1$ the statement follows that the properties of a function under rotation and inversion are preserved under $\mathbb{F}_{(N)}$. In its full generality, the specification of "properties under rotation" requires group theory. (In three dimensions, knowledge of spherical harmonics is required, while for $N = 2$, Fourier series is all one needs. This case will be developed in Section 8.4.) One property, *invariance*, is nevertheless easy to state: a function $f(\mathbf{q})$, $\mathbf{q} \in \mathscr{R}^N$, is invariant under orthogonal transformations if $f(\mathbf{q}) = f(\mathbf{A}^{-1}\mathbf{q})$ for all orthogonal $\mathbf{A}$. This means that the function can depend only on the norm $q := (\mathbf{q}\cdot\mathbf{q})^{1/2}$. Under Fourier transformation this property becomes $\tilde{f}(\mathbf{p}) = \tilde{f}(\mathbf{A}^{-1}\mathbf{p})$, so $\tilde{f}$ in turn can also only depend on $p := (\mathbf{p}\cdot\mathbf{p})^{1/2}$.

8.3.3. The Diffusive–Elastic Medium with Sources

We shall illustrate the use of the N-dimensional Fourier transform in finding the general form for the solution of the elastic, diffusive medium with sources in N dimensions, governed by

$$\nabla^2 f(\mathbf{q}, t) + F(\mathbf{q}, t) = c^{-2}\frac{\partial^2}{\partial t^2} f(\mathbf{q}, t) + a^{-2}\frac{\partial}{\partial t} f(\mathbf{q}, t), \tag{8.45}$$

with initial conditions $f(\mathbf{q}, t_0)$ and $\dot{f}(\mathbf{q}, t_0) := \partial f(\mathbf{q}, t)/\partial t|_{t=t_0}$ at some "initial" time t_0. Expression (8.45) resembles in part the diffusion equation, Eq. (5.1), with diffusion constant a, and in part the wave equation (5.15) with propagation velocity c. The sum of the two terms on the right-hand side states that the acceleration of the observable f due to its curvature at $\mathbf{q}$ is *diminished* by the velocity-dependent term, which has the effect of a viscous braking force. Further, the source term $F(\mathbf{q}, t)$ acts as a driving force in the regions of $(\mathbf{q}, t)$-space where it applies. The limits $c \to \infty$ and $a \to \infty$ lead, respectively, to the simple heat and wave equations in N dimensions.

Assuming $f(\mathbf{q}, t)$ and its derivatives are square-integrable in $\mathscr{R}^N$, we can apply the N-dimensional Fourier transform to (8.45), obtaining

$$-p^2\tilde{f}(\mathbf{p}, t) + \tilde{F}(\mathbf{p}, t) = c^{-2}\frac{\partial^2}{\partial t^2}\tilde{f}(\mathbf{p}, t) + a^{-2}\frac{\partial}{\partial t}\tilde{f}(\mathbf{p}, t), \tag{8.46}$$

with initial conditions $\tilde{f}(\mathbf{p}, t_0) := [\mathbb{F}_{(N)}\mathbf{f}(\cdot, t_0)](\mathbf{p})$ and $\dot{\tilde{f}}(\mathbf{p}, t_0)$. In this equation the most difficult part, the N-dimensional Laplacian operator, has been converted into a factor of $-p^2$ as with one-dimensional problems. Equation (8.46) is thus a second-order ordinary differential equation in time, which has been solved in Section 7.3 [Eq. (7.111)] using Fourier transforms and again in Section 8.1 [Eq. (8.16)] using Laplace techniques. By treating p^2 as a parameter and establishing the correspondence between $\{c^{-2}, a^{-2}, p^2, t\}$ and $\{M, c, k, q\}$, the solution to (8.46) is

$$\tilde{f}(\mathbf{p}, t) = \tilde{f}_F(\mathbf{p}, t) + \tilde{f}_B(\mathbf{p}, t). \tag{8.47}$$

The *transient* term solution to the homogeneous equation (8.46) (with the $\tilde{F}$ term absent) is given in terms of the boundary conditions at time t_0 as

$$\begin{aligned}\tilde{f}_B(\mathbf{p}, t) = {} & \tilde{f}(\mathbf{p}, t_0)[a^{-2}\tilde{G}(\mathbf{p}, t - t_0) + c^{-2}\dot{\tilde{G}}(\mathbf{p}, t - t_0)] \\ & + \dot{\tilde{f}}(\mathbf{p}, t_0)\tilde{G}(\mathbf{p}, t - t_0).\end{aligned} \tag{8.48}$$

The function $\tilde{G}(\mathbf{p}, t - t_0)$ can be copied from the simple oscillator Green's function [Eq. (2.11a), (2.11b), (2.12), (2.13a), (2.13b), (7.116), or (8.21), exchanging symbols as before and ω_e for p_e] as

$$\tilde{G}(\mathbf{p}, t) = \begin{cases} c^2 \exp(-\Gamma t) \sin \omega_e t/\omega_e, & t > 0, \\ 0, & t \leqslant 0, \end{cases} \tag{8.49a}$$

$$\Gamma := c^2/2a^2, \qquad \omega_e := (c^2p^2 - \Gamma^2)^{1/2}. \tag{8.49b}$$

The time derivative of (8.49) is $\dot{\tilde{G}}(\mathbf{p}, t)$. The tilde has been kept for (8.49) as its $\mathbb{F}_{(N)}^{-1}$ transform will be the Green's function for the equation (8.45) we are solving.

The *stationary* solution to (8.46) due to the source $\tilde{F}(\mathbf{p}, t)$ is given by the *convolution*—with respect to t—of the source function with (8.49) [see (7.117) or (8.22) with the proper symbol exchange], i.e.,

$$\tilde{f}_F(\mathbf{p}, t) = (\tilde{F} \overset{t}{*} \tilde{G})(\mathbf{p}, t) = \int_{t_0}^{t} dt' \tilde{F}(\mathbf{p}, t')\tilde{G}(\mathbf{p}, t - t'). \tag{8.50}$$

We assume the source to start operating not earlier than the initial time t_0.

Equation (8.45), whose solution we are seeking, now requires that we apply $\mathbb{F}_{(N)}^{-1}$ to (8.47)–(8.50). The transient term (8.48) is the product of functions of $\mathbf{p}$; hence its inverse transform will be a convolution over $\mathbf{q}$ of the factors. As the initial conditions are assumed to be given, the key lies in finding

$$\begin{aligned} G_N(\mathbf{q}, t) &:= [\mathbb{F}_{(N)}^{-1}\tilde{G}(\cdot, t)](\mathbf{q}) \\ &= (2\pi)^{-N/2}c^2 \exp(-\Gamma t) \int_{\mathscr{R}^N} d^N\mathbf{p}(c^2p^2 - \Gamma^2)^{-1/2} \\ &\quad \times \sin[(c^2p^2 - \Gamma^2)^{1/2}t] \exp(i\mathbf{p}\cdot\mathbf{q}). \end{aligned} \tag{8.51}$$

Once this Green's function—and its time derivative—is found, the transient term will be given by

$$\begin{aligned} f_B(\mathbf{q}, t) &= (2\pi)^{-N/2}\{f(\cdot, t_0) \overset{q}{*} [a^{-2}G_N(\cdot, t - t_0) + c^{-2}\dot{G}_N(\cdot, t - t_0)]\}(\mathbf{q}) \\ &\quad + (2\pi)^{-N/2}c^{-2}[\dot{f}(\cdot, t_0) \overset{q}{*} G_N(\cdot, t - t_0)](\mathbf{q}). \end{aligned} \tag{8.52a}$$

The stationary solution will be a *double* convolution—with respect to $\mathbf{q}$ and t—of the source with the Green's function:

$$\begin{aligned} f_F(\mathbf{q}, t) &= (2\pi)^{-N/2}(F ** G_N)(\mathbf{q}, t) \\ &= (2\pi)^{-N/2} \int_{t_0}^{t} dt' \int_{\mathscr{R}^N} d^N\mathbf{q}' F(\mathbf{q}', t')G_N(\mathbf{q} - \mathbf{q}', t - t'). \end{aligned} \tag{8.52b}$$

The most general solution will be, finally, (8.52a) plus (8.52b).

8.3.4. Wave Equation in Three Dimensions

As before, it will simplify matters to look for the *fundamental* solutions to the equation of motion, that is, those solutions or their time derivatives which at $t = t_0$ are Dirac δ's in $\mathbf{q}$, as these are found in terms of the Green's function $G_N(\mathbf{q}, t - t_0)$ and its time derivative. In this section we shall examine two limiting cases of interest: the wave equation in three dimensions, obtained in the limit $a \to \infty$, and the diffusion equation in N dimensions, which is the limit $c \to \infty$. The wave equation for two and N dimensions and the general solution of (8.45) will be found in Section 8.4. We shall attach the indices w for wave and h for heat to the Green's functions in order to avoid confusion.

The wave equation in three dimensions simplifies the problem of finding the Green's function (8.51) since for $a \to \infty$, $\Gamma \to 0$. We are left with the calculation of the three-dimensional inverse Fourier transform of $\tilde{G}^w(\mathbf{p}, t) = c \sin(cpt)/p$. By introducing the well-known spherical coordinates

$$\begin{aligned} p_1 &= p \sin\theta \sin\varphi, & p &\in [0, \infty), \\ p_2 &= p \sin\theta \cos\varphi, & \theta &\in [0, \pi], \\ p_3 &= p \cos\theta, & \varphi &\in [0, 2\pi), \end{aligned} \tag{8.53}$$

and setting the $\theta = 0$ direction along the vector $\mathbf{q}$ so that $\mathbf{p}\cdot\mathbf{q} = pq\cos\theta$, the volume integrals (8.38) become

$$\int_{\mathscr{R}^3} d^3\mathbf{p} \cdots = \int_0^\infty p^2\, dp \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\varphi \cdots. \tag{8.54}$$

We thus calculate

$$\begin{aligned} G_3{}^w(\mathbf{q}, t) &= (2\pi)^{-3/2} c \int_0^\infty p\, dp \sin(cpt) \int_0^\pi \sin\theta\, d\theta \exp(-ipq\cos\theta) \int_0^{2\pi} d\varphi \\ &= (2\pi)^{-1/2} c \int_0^\infty p\, dp \sin(cpt) \int_{-1}^1 du \exp(-ipqu) \qquad (u := \cos\theta) \\ &= (2\pi)^{-1/2}(2c/q) \int_0^\infty dp \sin(cpt) \sin pq \\ &= (2\pi)^{-1/2}(c/q) \int_0^\infty dp\{\cos[p(q - ct)] - \cos[p(q + ct)]\} \\ &= (2\pi)^{1/2}(c/2q)[\delta(q - ct) - \delta(q + ct)] \qquad \text{[Eq. (7.93)]} \\ &= (2\pi)^{1/2} c\,\delta(\mathbf{q}^2 - c^2t^2) \qquad \text{[Eq. (7.94b)]}. \end{aligned} \tag{8.55}$$

Similarly, we find

$$\dot{G}_3{}^w(\mathbf{q}, t) = -(2\pi)^{1/2}(2c^2/q)[\delta'(q - ct) + \delta'(q + ct)]. \tag{8.56}$$

The general solution to initial conditions is thus found from (8.48) (for $a \to \infty$), (8.55), and (8.56) for $t > t_0$, and by remembering that $q \geqslant 0$,

$$\begin{aligned} f_B(\mathbf{q}, t) = &-(4\pi)^{-1} \int_{\mathscr{R}^3} d^3\mathbf{q}' f(\mathbf{q}', t_0) |\mathbf{q} - \mathbf{q}'|^{-1} \delta'(|\mathbf{q} - \mathbf{q}'| - c(t - t_0)) \\ &+ (4\pi c)^{-1} \int_{\mathscr{R}^3} d^3\mathbf{q}' \dot{f}(\mathbf{q}', t_0) |\mathbf{q} - \mathbf{q}'|^{-1} \delta(|\mathbf{q} - \mathbf{q}'| - c(t - t_0)). \end{aligned} \tag{8.57}$$

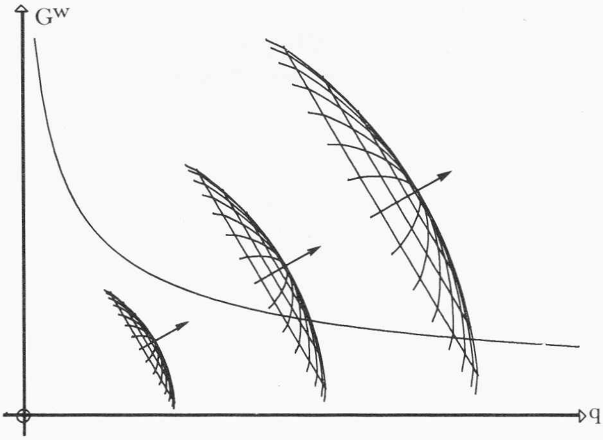


Fig. 8.8. The fundamental solutions $G_3{}^w(\mathbf{q}, t)$ of the three-dimensional wave equation are expanding spherical singularity shells modulated by a radial geometric factor of q^{-1} (continuous line).

It should be observed that wave propagation in three-dimensional elastic media has the following properties:

(a) The *causality principle* is obeyed due to the appearance of the δ's and the restriction $t > t_0$: if a disturbance is localized at a point $\mathbf{q}_0$ at time t_0, no information is available at a point $\mathbf{q}_1$ as long as $|\mathbf{q}_0 - \mathbf{q}_1| > c(t - t_0)$. [Just for the record, it should be noted that (8.55) and (8.56) possess *advanced* solutions for $t < t_0$ besides the *retarded* ones for $t > t_0$ which were kept in (8.57); the former are usually considered nonexistent based on the present lack of solid experimental evidence. Yet see Puthoff and Targ (1976, Sections IV and V and the references within).]

(b) *Reciprocity* holds. The effect of a disturbance at $\mathbf{q}_0$ on $\mathbf{q}_1$ is the same as that of a disturbance at $\mathbf{q}_1$ on $\mathbf{q}_0$ if their proper time ordering is respected. This is a consequence of our assumption that space is homogeneous and isotropic—Eq. (8.45) involves only ∇^2—and is reflected in the fact that the Green's function (8.57) is a function of $|\mathbf{q} - \mathbf{q}'|$ only.

(c) A point (singular) disturbance at t_0 propagates as an expanding spherical singularity shell of radius $c(t - t_0)$ modulated by a *geometric* factor $|\mathbf{q}|^{-1}$. See Fig. 8.8. This factor gives rise to the familiar "inverse-square" law for isotropic illumination, the latter being proportional to the square of the disturbance amplitude.

(d) There is no backwave; once the expanding singularity shell described above passes over a point, the medium again remains at rest.

Exercise 8.16. Consider the case of $N = 1$ dimension. This only simplifies the necessary inverse Fourier transformation. The Green's functions obtained will be identical to (5.27) found for the case of elastic media with fixed ends. Except for the lifting of this restriction, all conclusions of Section 5.2 continue to hold. Note carefully that the three-dimensional Green's functions (8.55)–(8.56) are $-q^{-1}\partial/\partial q$ times the one-dimensional Green's functions (5.27a)–(5.27b). This fact will be generalized in Section 8.4.

Exercise 8.17. Consider the *energy* in a three-dimensional elastic vibrating medium. This can be found along lines parallel to (5.40)–(5.42), except that integration proceeds over $\mathscr{R}^3$. Show that the *partial energy* of each constituent wave $E_{\mathbf{p}} := p^2|\tilde{f}(\mathbf{p}, t)|^2 + c^{-2}|\dot{\tilde{f}}(\mathbf{p}, t)|^2$ is separately conserved. As the medium is governed by a linear equation, there will be no energy exchange between different partial waves. The wave equation thus has a continuous infinity of conservation laws, one for each value of **p**.

Exercise 8.18. Propose solutions to the three-dimensional wave equation of the form $(2\pi)^{-3/2} \exp[i(\mathbf{p}\cdot\mathbf{q} + pct)]$. Expand a general solution in terms of these and the partial-wave coefficients in terms of the initial conditions. Thus reconstitute Eq. (8.57). [See the article by Halevi (1973).]

Exercise 8.19. Show the transitivity of time evolution. Compare with Exercises 5.10 and 5.17.

8.3.5. The Diffusion Equation in N Dimensions

A second family of cases where Eq. (8.45) yields to an easy solution is the limit $c \to \infty$, when the medium becomes purely diffusive. Although $\Gamma \to \infty$ and $\omega_e \simeq i(\Gamma - a^2p^2)$, the limit of (8.49a) is well defined. It is $\tilde{G}_N{}^h(\mathbf{p}, t) = a^2 \exp(-a^2\mathbf{p}^2 t)$. The inverse $\mathbb{F}_N$ transform of $\tilde{G}_N{}^h(\mathbf{p}, t)$ is easy to find in Cartesian coordinates, the function being the product of Gaussians of width $(2a^2t)^{-1}$ in each of the coordinates. By Eq. (7.22) we find, in N dimensions,

$$\begin{aligned} G_N{}^h(\mathbf{q}, t) &= a^2 \prod_{k=1}^{N} (\pi/a^2t)^{1/2}[\mathbb{F}^{-1}\mathbf{G}_{(2a^2t)^{-1}}](q_k) \\ &= a^2(2a^2t)^{-N/2} \exp(-\mathbf{q}^2/4a^2t). \end{aligned} \tag{8.58}$$

The general transient solution thus becomes, from (8.52a),

$$f_B(\mathbf{q}, t) = (4\pi a^2 t)^{-N/2} \int_{\mathscr{R}^N} d^N\mathbf{q}' f(\mathbf{q}', t_0) \exp[-(\mathbf{q} - \mathbf{q}')^2/4a^2(t - t_0)], \tag{8.59}$$

leaving out the initial velocity as a boundary condition for $f_B(\mathbf{q}, t)$. The differential equation is now of first order in time. The fundamental solutions are spreading Gaussians of width $2a^2(t - t_0)$ with a decreasing maximum of $(4\pi a^2t)^{-N/2}$. See Fig. 8.9. One can see that the temperature maximum drops faster for higher dimensions: heat simply has more directions in which to escape.

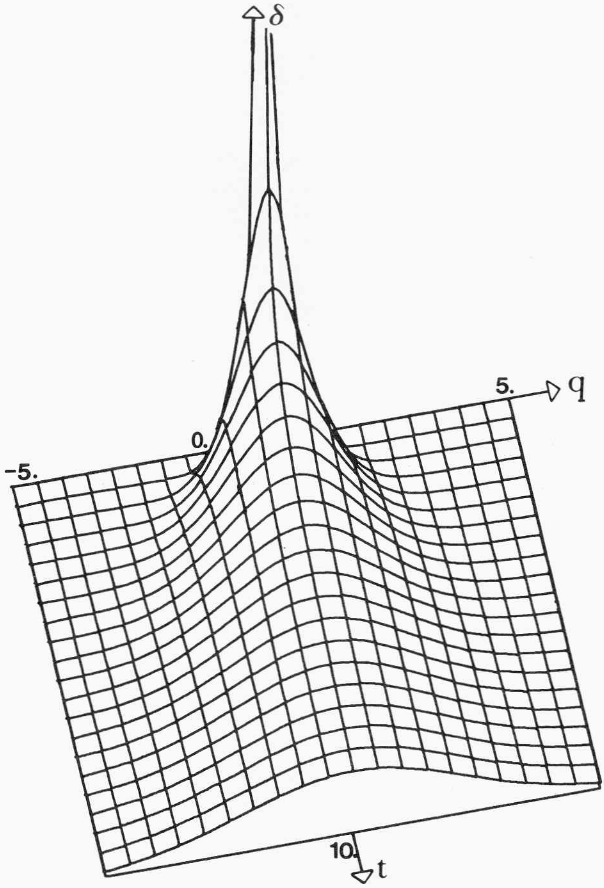


Fig. 8.9. The fundamental solution $G_1^h(q, t)$ of the one-dimensional diffusion equation (for $a = \frac{1}{2}$). The initial condition given by a Dirac δ at $q = 0$ develops in time as a spreading Gaussian.

It is interesting to compare the Green's function for a one-dimensional heat flow in an unbounded medium (Fig. 8.9) with the heat flow in a *ring* discussed in Section 5.1 (the Green's function is shown in Fig. 4.13, reading time development upward). Exercise 8.20 indicates some further developments.

Exercise 8.20. Solve the N-dimensional homogeneous diffusion equation (8.45) ($c \to \infty$) by proposing separable solutions in all coordinates. Choosing the boundary conditions, you should arrive at (8.59).

Exercise 8.21. Prove that total heat is conserved. Compare with Exercise 5.1.

Table 8.5 A Function and Its N-Dimensional Fourier Transform under Various Operators and Operations[a]

Operation	$f(\mathbf{q})$	$\tilde{f}(\mathbf{p})$
Translation	$f(\mathbf{q} + \mathbf{y})$	$\exp(i\mathbf{p}\cdot\mathbf{y})\tilde{f}(\mathbf{p})$
	$\exp(-i\mathbf{q}\cdot\mathbf{x})f(\mathbf{q})$	$\tilde{f}(\mathbf{p} + \mathbf{x})$
Linear transformation	$f(\mathbf{A}\mathbf{q})$	$\lvert\det \mathbf{A}\rvert^{-1}\tilde{f}(\mathbf{A}^{T-1}\mathbf{p})$
Multiplication	$f(\mathbf{q})g(\mathbf{q})$	$(2\pi)^{-N/2}(\tilde{f} * \tilde{g})(\mathbf{p})$
Convolution	$(f * g)(\mathbf{q})$	$(2\pi)^{N/2}\tilde{f}(\mathbf{p})\tilde{g}(\mathbf{p})$
Differentiation	$\nabla f(\mathbf{q})$	$i\mathbf{p}\tilde{f}(\mathbf{p})$
	$-i\mathbf{q}f(\mathbf{q})$	$\nabla\tilde{f}(\mathbf{p})$

[a] Compare with Table 7.1 for the $N = 1$ case.

Exercise 8.22. Prove the transitivity of the time evolution given by (8.59). This is the analogue of Eq. (5.11) or (5.14). Equation (7.50) should come in very handy.

The description of the wave and diffusion phenomena as well as some generalizations such as the *telegraph* equation can be found in several theoretical physics texts. Notably, Courant and Hilbert (1962) dedicate several sections of Vol. 2 to these problems, using several solution methods in Chapters III and VI.

8.4. Hankel Transforms

If a system "looks the same" from any direction in space, we say that it is invariant under rotations or isotropic. This is the case, for instance, of gravitational attraction between point masses. It is also true of many potentials in spinless quantum mechanics. The isotropy of the system implies that the governing equations of motion depend only on rotationally invariant quantities such as functions of $q \coloneqq |\mathbf{q}| = (\mathbf{q}\cdot\mathbf{q})^{1/2}$ or derivatives as ∇^2. This was the case of the N-dimensional elastic–diffusive medium described by Eq. (8.45), which was not only isotropic but *homogeneous*: invariant under translations (only ∇^2 appears). The Fourier transform of an isotropic differential equation is itself isotropic, and thus the Green's function is a function of q only. Of course, initial conditions need not be isotropic. Only the laws of motion are. This brings us to examine more closely the N-dimensional Fourier transform of functions of the radial variable q only and, later, that of eigenfunctions of the rotation operators. Parametrizing N-dimensional space conveniently in spherical coordinates, we shall reduce the N-fold $\mathbb{F}_{(N)}$ integration to a single integral defining the Hankel transform.

8.4.1. Spherical Coordinates in N Dimensions

The problem of introducing spherical coordinates into N-dimensional space can be tackled by guiding ourselves with the two- and three-dimensional cases [Eqs. (6.14) and (8.53)]. Consider a vector of length q along the Nth coordinate $\mathbf{v}_N \coloneqq \{0, \ldots, 0, q\}$. (To save space we write column vectors as row vectors between braces.) We transform this vector by a rotation by an angle θ_{N-1} in the N–$(N-1)$ coordinate plane. The vector is then transformed into $\mathbf{v}_{N-1} \coloneqq \{0, \ldots, 0, q \sin\theta_{N-1}, q\cos\theta_{N-1}\}$. If $N = 2$, this is all we need to do. If $N > 2$, we now rotate $\mathbf{v}_{N-1}$ into the next higher subspace, through an angle θ_{N-2} in the $(N-1)$–$(N-2)$-plane, obtaining

$$\mathbf{v}_{N-2} \coloneqq \{0, \ldots, 0, q\sin\theta_{N-1}\sin\theta_{N-2}, q\sin\theta_{N-1}\cos\theta_{N-2}, q\cos\theta_{N-1}\}.$$

If $N = 3$, this is all. [See Eqs. (8.53).] If $N > 3$, we rotate through an angle θ_{N-3} in the $(N-2)$–$(N-3)$-plane and continue in this way, piling sines and cosines of the new angles on the components of the vector $\mathbf{v}_{N-k}$. Once a cosine is added to a component, it receives no new factors. The last rotation through θ_1 is in the 2–1 plane. The components of $\mathbf{v}_1$ are then, finally,

$$\begin{aligned}
q_1 &= q\sin\theta_{N-1}\sin\theta_{N-2}\cdots\sin\theta_2\sin\theta_1,\\
q_2 &= q\sin\theta_{N-1}\sin\theta_{N-2}\cdots\sin\theta_2\cos\theta_1,\\
q_3 &= q\sin\theta_{N-1}\sin\theta_{N-2}\cdots\cos\theta_2,\\
&\cdots\\
q_k &= q\sin\theta_{N-1}\cdots\sin\theta_k\cos\theta_{k-1},\\
&\cdots\\
q_{N-1} &= q\sin\theta_{N-1}\cos\theta_{N-2},\\
q_N &= q\cos\theta_{N-1}.
\end{aligned} \tag{8.60a}$$

If we let $\theta_{N-1} \in [0, \pi]$, then $q_N \in [-q, q]$, while the component q_{N-1} will take values in $[-q, q]$ when θ_{N-2} is also allowed to range over $[0, \pi]$—and similarly for θ_{N-3}, etc., up to θ_2. Last, θ_1 must range in $[0, 2\pi)$ if q_1 is to take positive as well as negative values. Hence, the angle ranges in (8.60a) are appropriately described by

$$\theta_1 \in [0, 2\pi), \qquad \theta_k \in [0, \pi], \qquad k = 2, 3, \ldots, N-1, q \in [0, \infty). \tag{8.60b}$$

For *any* N-dimensional vector $\mathbf{q}$ of components $\{q_1, q_2, \ldots, q_N\}$ we can find values of q and θ_k, $k = 1, 2, \ldots, N-1$, to parametrize its components. To find θ_k we construct

$$r_k \coloneqq (q_1^2 + q_2^2 + \cdots + q_k^2)^{1/2} = q\sin\theta_{N-1}\cdots\sin\theta_k = r_{k+1}\sin\theta_k, \tag{8.61a}$$

$$q_{k+1} = r_{k+1}\cos\theta_k, \tag{8.61b}$$

thus finding θ_k as $\arctan(r_k/q_{k+1})$ for $k = 1, 2, \ldots, N-1$, $r_1 = q_1$, and

$r_N = q$. Equations (8.60) thus serve to *define* spherical coordinates for N-space. We can find the *volume element* $d^N\mathbf{q}$ from Eqs. (8.61), since for fixed k they tell us that the two-dimensional vector $\{r_k, q_{k+1}\}$ is represented in polar coordinates as having radius r_{k+1} and angle θ_k as in (6.14b). Thus

$$dr_k\, dq_{k+1} = r_{k+1}\, dr_{k+1}\, d\theta_k, \qquad k = 1, 2, \ldots, N-1. \tag{8.62a}$$

Hence, recursively,

$$\begin{aligned} d^N\mathbf{q} &= dq_N\, dq_{N-1} \cdots dq_3\, dq_2\, dq_1 \\ &= dq_N \cdots dq_3 r_2\, dr_2\, d\theta_1 \\ &= dq_N \cdots dq_4 r_2 r_3\, dr_3\, d\theta_2\, d\theta_1 = \cdots \\ &= r_2 r_3 \cdots r_N\, dr_N\, d\theta_{N-1} \cdots d\theta_2\, d\theta_1 \\ &= q^{N-1}\, dq \sin^{N-2}\theta_{N-1}\, d\theta_{N-1} \cdots \sin^{N-k-1}\theta_{N-k}\, d\theta_{N-k} \cdots d\theta_1. \end{aligned} \tag{8.62b}$$

This allows us to calculate the $(N-1)$-dimensional surface of the sphere S_{N-1} in N dimensions as

$$\begin{aligned} |S_{N-1}| &= \int_0^\pi \sin^{N-2}\theta_{N-1}\, d\theta_{N-1} \cdots \int_0^\pi \sin\theta_2\, d\theta_2 \int_0^{2\pi} d\theta_1 \\ &= \{\pi^{1/2}\Gamma((N-1)/2)/\Gamma(N/2)\}|S_{N-2}|, \end{aligned} \tag{8.63a}$$

where we have used the Wallis integral for $\sin^m\theta$. Since $|S_1| = 2\pi$,

$$|S_{N-1}| = 2\pi^{N/2}/\Gamma(N/2). \tag{8.63b}$$

We verify that $|S_2| = 4\pi$ is the 2-surface of the usual sphere in three dimensions.

8.4.2. Reduction of the Fourier to the Hankel Transform

We can now tackle the problem of finding the N-dimensional Fourier transform of a function $f(q)$ of the radial variable q. Choosing the $\theta_{N-1} = 0$ axis along the direction of $\mathbf{p}$ so that $\mathbf{p}\cdot\mathbf{q} = pq\cos\theta_{N-1}$, we must perform

$$\begin{aligned} \tilde{f}(p) &= (2\pi)^{-N/2} \int_{\mathscr{R}^N} d^N\mathbf{q} f(q) \exp(-ipq\cos\theta_{N-1}) \\ &= (2\pi)^{-N/2}|S_{N-2}| \int_0^\infty q^{N-1}\, dq f(q) \int_0^\pi \sin^{N-2}\theta_{N-1}\, d\theta_{N-1} \\ &\quad \times \exp(-ipq\cos\theta_{N-1}). \end{aligned} \tag{8.64}$$

The integrations over the angles $\theta_{N-2}, \ldots, \theta_1$ have yielded $|S_{N-2}|$ as given by (8.63), and we are left with a single integral over θ_{N-1} of the type

$$\int_0^\pi \sin^{2\mu}\theta\, d\theta \exp(\pm iz\cos\theta) = \pi^{1/2}\Gamma(\mu + 1/2)(z/2)^{-\mu}J_\mu(z). \tag{8.65}$$

The Bessel function (Appendix B) thus enters into the picture. Substituting (8.65) into (8.64) and canceling the Γ-functions, we find

$$\tilde{f}(p) = p^{-N/2+1} \int_0^\infty q^{N/2}\, dq f(q) J_{N/2-1}(pq) =: (\mathbb{H}^B_{N/2-1}\mathbf{f})(p), \quad (8.66a)$$

which is defined as the *Hankel–Bochner* transform of order $N/2 - 1$ of $f(q)$. The inverse transform follows with only a change of sign in the $\exp(-i\mathbf{p}\cdot\mathbf{q})$ factor, rendered innocuous by the double sign in (8.65), so that

$$f(q) = q^{-N/2+1} \int_0^\infty p^{N/2}\, dp \tilde{f}(p) J_{N/2-1}(pq) =: (\mathbb{H}^{B-1}_{N/2-1}\tilde{\mathbf{f}})(q). \quad (8.66b)$$

Notice that the transform kernels of $\mathbb{H}_\mu{}^B$ and $\mathbb{H}_\mu^{B-1}$ are the same, and hence $\mathbb{H}_\mu^{B2} = \mathbb{1}$. [Compare with the property of the Fourier transform, where $\mathbb{F}^2 = \mathbb{I}_0$; see Eq. (7.25).] This is to be expected, as from the $\mathbb{F}_{(N)}$ point of view we are dealing with rotationally invariant functions. These functions are *even* in each of the Cartesian coordinates, and thus $\mathbb{I}_0$ is equivalent to $\mathbb{1}$ in their subspace.

A Parseval formula holds for the Hankel–Bochner transforms (8.66a)–(8.66b):

$$\begin{aligned}(\mathbf{f}, \mathbf{g})_N &= \int_{\mathscr{R}^N} d^N\mathbf{q} f(q)^* g(q) = |S_{N-1}| \int_0^\infty q^{N-1}\, dq f(q)^* g(q) \\ &= (\tilde{\mathbf{f}}, \tilde{\mathbf{g}})_N = |S_{N-1}| \int_0^\infty p^{N-1}\, dp \tilde{f}(p)^* \tilde{g}(p). \end{aligned} \quad (8.67)$$

Exercise 8.23. Examine the Hankel–Bochner transform (8.66) for $N = 1$ dimension. Show that we are dealing with *even* functions and the *cosine* Fourier transform, as $J_{-1/2}(z) = (2/\pi z)^{1/2} \cos z$.

Exercise 8.24. Examine the Hankel–Bochner transform for $N = 3$ dimensions. This has already been used in Section 8.3. In fact, it reduces to the *sine* Fourier transform as $J_{1/2}(z) = (2/\pi z)^{1/2} \sin z$.

Exercise 8.25. Find the Hankel–Bochner transform of degree μ of $f(cr)$ in terms of that of $f(r)$.

8.4.3. Recursion Relations

There is a relation between Hankel–Bochner transforms of orders μ and $\mu + m$, m an integer. This comes from a recursion relation for Bessel functions, which can be found directly from (8.66a) and (8.65) for $\mu = N/2 - 1$:

$$-\frac{1}{p}\frac{d}{dp}(\mathbb{H}_\mu{}^B\mathbf{f})(p) = (2\pi)^{-\mu-1}|S_{2\mu}| \int_0^\infty q^{2\mu+1}\,dq f(q)$$
$$\times \int_0^\pi \sin^{2\mu}\theta\, d\theta (ip^{-1}q\cos\theta)\exp(-ipq\cos\theta)$$
$$= (2\pi)^{-\mu-1}|S_{2\mu}| \int_0^\infty q^{2\mu+1}\,dq f(q)(ip^{-1}q)(2\mu+1)^{-1}$$
$$\times \left[\sin^{2\mu+1}\theta\exp(-ipq\cos\theta)\Big|_{\theta=0}^{\pi} - ipq\int_0^\pi \sin^{2\mu+2}\theta\, d\theta\exp(-ipq\cos\theta)\right]$$
$$= (\mathbb{H}_{\mu+1}^B\mathbf{f})(p), \tag{8.68}$$

where we have used integration by parts and the recursion (8.63a). It follows that

$$(-p^{-1}\,d/dp)^m(\mathbb{H}_\mu{}^B\mathbf{f})(p) = (\mathbb{H}_{\mu+m}^B\mathbf{f})(p), \tag{8.69}$$

which relates the Hankel–Bochner transforms of orders differing by an integer.

8.4.4. Odd- and Even-Dimensional Wave Equations

As the Hankel–Bochner transform of degree $\mu = N/2 - 1$ of a function $f(q)$ is the N-dimensional Fourier transform of the function $f(q)$ of radius q, we can immediately put Eq. (8.69) to work on the problem—stated in Section 8.3—of finding the Green's function for the N-dimensional wave equation. This will show some of the characteristics of the solutions for general N. From (8.51) with $\Gamma = 0$ and $\tilde{G}^w(\mathbf{p}, t) = c\sin cpt/p$, we have

$$G_N{}^w(\mathbf{q}, t) = [\mathbb{F}_{(N)}\tilde{\mathbf{G}}^w(\cdot, t)](\mathbf{q}) = [\mathbb{H}_{N/2-1}^B\tilde{\mathbf{G}}^w(\cdot, t)](q)$$
$$= (-q^{-1}\,\partial/\partial q)^m[\mathbb{H}_{(N-2m)/2-1}^B\tilde{\mathbf{G}}^w(\cdot, t)](q)$$
$$= (-q^{-1}\,\partial/\partial q)^m G_{N-2m}^w(q, t) = [-2\,\partial/\partial(q^2)]^m G_{N-2m}^w(q, t). \tag{8.70}$$

In (8.55) and (8.56) we have calculated $G_3{}^w(\mathbf{q}, t)$, and so we have the expressions for odd dimension $2n + 3$. Keeping only the retarded solution, we have

$$G_{2n+3}^w(q, t) = (-q^{-1}\,\partial/\partial q)^n G_3{}^w(q, t)$$
$$= c(\pi/2)^{1/2}(-q^{-1}\,\partial/\partial q)^n[q^{-1}\delta(q - ct)], \tag{8.71a}$$
$$\dot{G}_{2n+3}^w(q, t) = -c^2(\pi/2)^{1/2}(-q^{-1}\,\partial/\partial q)^n[q^{-1}\delta'(q - ct)], \tag{8.71b}$$

which represents an expanding singularity shell. It exhibits (a) causality,

(b) reciprocity, (c) a leading modulation factor q^{-n}, and (d) no backwave or wake.

Exercise 8.26. Verify that (8.71) correctly relates the results for one- and three-dimensional spaces. See again Exercise 8.16.

To find the wave equation Green's function in an even number of dimensions we can produce a two-dimensional world out of a three-dimensional one by assuming that all relevant objects are cylinders along the q_3-axis, that is, the initial conditions are independent of q_3. In doing this, the integral over q_3 in the convolution (8.57) can be performed on the Green's function alone, i.e.,

$$H_2^w(\mathbf{q}\{q_1, q_2\}, t) := \int_{-\infty}^{\infty} dq_3 G_3^w(\mathbf{q}\{q_1, q_2, q_3\}, t). \tag{8.72a}$$

We can perform this in Cartesian coordinates for the retarded part of (8.55):

$$\begin{aligned} H_2^w(\mathbf{q}\{q_1, q_2\}, t) &= c(\pi/2)^{1/2} \int_{-\infty}^{\infty} dq_3 (q_1^2 + q_2^2 + q_3^2)^{-1/2} \delta((q_1^2 + q_2^2 + q_3^2)^{1/2} - ct) \\ &= c(\pi/2)^{1/2} \int_{-\infty}^{\infty} dq_3 (q_1^2 + q_2^2 + q_3^2)^{-1/2} ct (c^2t^2 - q_1^2 - q_2^2)^{-1/2} \\ &\quad \times \{\delta(q_3 - (c^2t^2 - q_1^2 - q_2^2)^{1/2}) + \delta(q_3 + (c^2t^2 - q_1^2 - q_2^2)^{1/2})\}, \end{aligned} \tag{8.72b}$$

where in the last equality we have used the expression (7.96) for a $\delta[F(q)]$ in terms of $\delta(q - a_i)$, a_i being the roots of $F(q)$; namely,

$$a_{1,2} = \pm(c^2t^2 - q_1^2 - q_2^2)^{1/2}$$

when $q_1^2 + q_2^2 < c^2t^2$. No roots exist for $q_1^2 + q_2^2 > c^2t^2$. Thus in a two-dimensional space,

$$G_2^w(\mathbf{q}, t) = c(c^2t^2 - q^2)^{-1/2}\Theta(ct - q) = G_2^w(q, t), \tag{8.73}$$

where we have introduced the Heaviside step function Θ, Eq. (7.89), and divided by $(2\pi)^{1/2}$, since the two-dimensional Green's function will be present in convolutions (8.52) with factors $(2\pi)^{-1}$ instead of $(2\pi)^{-3/2}$ as for three dimensions. As $\Theta'(q) = \delta(q)$ and $\Theta(0) := 1/2$, from (8.73) we find

$$\begin{aligned} \dot{G}_2^w(q, t) &= -c^3t(c^2t^2 - q^2)^{-3/2}\Theta(ct - q) + c^2(c^2t^2 - q^2)^{-1}\delta(q - ct) \\ &= [2c\delta(q - ct) - c^2t(c^2t^2 - q^2)^{-1}]G_2^w(q, t). \end{aligned} \tag{8.74}$$

Equations (8.73) and (8.74), as well as the Green's functions for a higher, even number of dimensions obtained by (8.71), fulfill the properties (a), (b), and (c), as do the solutions of the wave equation in other dimensions. As to property (d), the behavior is different. The disturbance, if originally localized, will develop a trailing wake, because of the non-δ form of $G_2{}^w(q, t)$, which *smears* any initial condition out of its original sharpness. This trailing wake, backwave or reverberation is a characteristic of all even-dimensional spaces. A stone thrown in a pond does not quite reproduce, thus, the behavior of waves in three-dimensional space.

8.4.5. General Solution of the Diffusive–Elastic Equation

For the Green's function of the N-dimensional general elastic–diffusive medium, we have to calculate, as in (8.70), the inverse Hankel transform of (8.49), namely,

$$\begin{aligned} G_N(\mathbf{q}, t) &= \big(\mathbb{F}_{(N)}\tilde{G}(\cdot, t)\big)(\mathbf{q}) = \big(\mathbb{H}^B_{N/2-1}\tilde{G}(\cdot, t)\big)(q) \\ &= c^2 q^{-N/2+1} \exp(-\Gamma t) \int_0^\infty dp p^{N/2} (c^2p^2 - \Gamma^2)^{1/2} \\ &\quad \times \sin[t(c^2p^2 - \Gamma^2)^{1/2}] J_{N/2-1}(pq). \end{aligned} \tag{8.75}$$

This is a rather difficult integral to do "by hand." It appears in the literature, however, as a particular case of the discontinuous Sonine integrals. [See Watson (1922, Section 13.47). In the tables of Hankel transforms by Oberhettinger (1973), it can be found by Eq. 6.43-II.] The result is

$$\begin{aligned} G_N(q, t) &= c(\pi/2)^{1/2}[\Gamma c^{-1}(c^2t^2 - q^2)^{-1/2}]^{(N-1)/2} \\ &\quad \times \exp(-\Gamma t) I_{-(N-1)/2}\big(\Gamma c^{-1}(c^2t^2 - q^2)^{1/2}\big)\Theta(ct - q), \\ &\qquad q \neq ct, \end{aligned} \tag{8.76}$$

where $I_\nu(z)$ is the *modified Bessel function* (see Appendix B) and Θ the usual Heaviside step. This function has been plotted for various values of parameters and variables in Fig. 8.10. The Green's function (8.76) includes δ terms at $q = ct$ for $N > 2$. These can be obtained for odd N using (8.68) and the fact that $G_1(q, t)$ is simply discontinuous at the advancing edge of the wave. For N even, one starts from $G_2(q, t)$.

Exercise 8.27. Consider the diffusion equation limit $c \to \infty$ of the Green's function (8.76), recalling that $I_\nu(z) \sim (2\pi z)^{-1/2} \exp(z)$ as $z \to \infty$. Verify that Eq. (8.58) is correctly reproduced.

Exercise 8.28. Consider the wave equation limit $a \to \infty$ of (8.76) using $I_\nu(z) \sim (z/2)^\nu/\Gamma(\nu + 1)$ for $\nu \neq -1, -2, \ldots$ as $z \to 0$. This verifies only the even-dimensional cases. Show that for $N = 2$, (8.73) is correctly given. For N odd and larger than 1, the result is zero since (8.76) holds for $q \neq ct$.

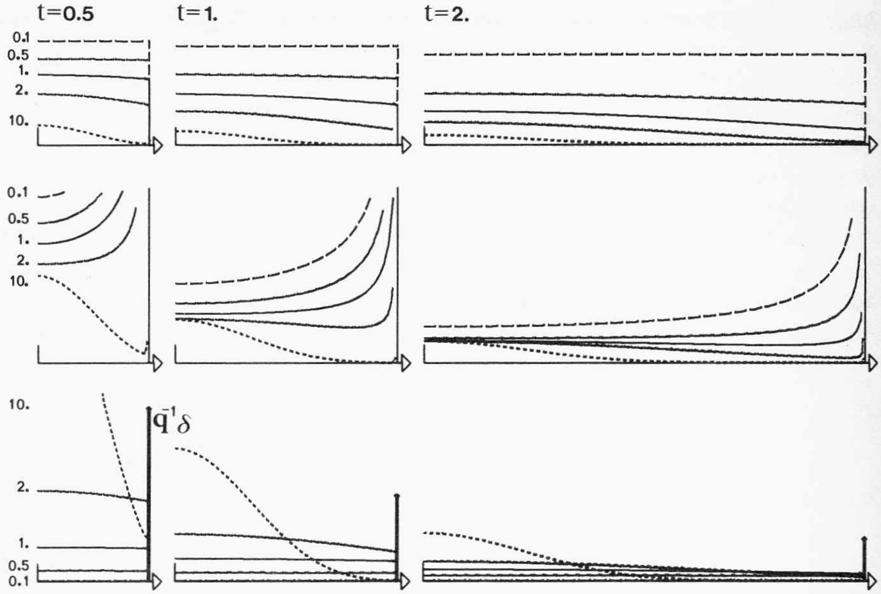


Fig. 8.10. Green's functions for elastic–diffusive media in one, two, and three dimensions (top, middle, and bottom rows). With advancing time ($t = 0.5$, 1, and 2 in the first, second, and third columns), the disturbance spreads up to $q = ct$ and is zero from there on. In each graph we have plotted the function for values of $\Gamma = 0.1$, 0.5, 1, 2, and 10 with different dottings. The first value corresponds nearly to the "wave" limit and the last value to the "diffusive" one. [Note that a change of scale is still needed: Eqs. (8.52).] The two- and three-dimensional cases have a singular edge: a $(c^2t^2 - q^2)^{-1/2}$ factor for two dimensions and a $q^{-1}\delta(q - ct)$ summand for three.

Exercise 8.29. Show that the Laplacian operator in N dimensions, Eq. (6.2), can be written in spherical coordinates (8.60) as

$$\nabla^2_{(N)} = r_N^{-N+1} \frac{\partial}{\partial r_N} r_N^{N-1} \frac{\partial}{\partial r_N} + r_N^{-2}\Lambda^2_{(N-1)}, \tag{8.77a}$$

where $r_N := q$, the radial coordinate in N-space, and $\Lambda^2_{(N-1)}$ is the *Laplacian on* S_{N-1},

$$\Lambda^2_{(k)} = \sin^{-k+2}\theta_{k-1} \frac{\partial}{\partial \theta_{k-1}} \sin^{k-2}\theta_{k-1} \frac{\partial}{\partial \theta_{k-1}} + \sin^{-2}\theta_{k-1}\Lambda^2_{(k-1)}, \tag{8.77b}$$

$$\Lambda^2_{(2)} = \partial^2/\partial\theta_1{}^2. \tag{8.77c}$$

This can be done recursively. For $N = 2$, Eqs. (8.77) match Eq. (6.16). If (8.77a)–(8.77b) are valid for $N = k$, $\nabla^2_{(k)}$ involving second derivatives with respect to $q_1, q_2, \ldots, q_k$, then they also hold for $\nabla^2_{(k+1)} = \partial^2/\partial q^2_{k+1} + \nabla^2_{(k)}$. Use the recursiveness provided by Eqs. (8.61), the two dimensional case for the (r_k, q_{k+1})-plane, and $r_k^{-1}\,\partial/\partial r_k = r_{k+1}^{-1}\,\partial/\partial r_{k+1} + r_{k+1}^{-2}\cot\theta_k\,\partial/\partial\theta_k$.

8.4.6. Hankel Transforms and Definite Symmetry under Rotations

Another context in which Hankel transforms arise is in finding the Fourier transforms of functions with *definite transformation properties* under rotations. This term merits some explanation. (In Section 4.3 we used the synonymous characterization of *definite symmetry* under translations on a circle.) We examine functions, or *sets* of functions, which transform among themselves under rotations. Their characterization is aided if we ask them to be eigenfunctions of a rotationally invariant self-adjoint operator, as then they will constitute complete and orthogonal sets of functions on the angle variables. In two dimensions the procedure can be implemented in terms of Fourier techniques. If we choose functions $f_m(\mathbf{q}) := f(q)\exp(im\theta_q)$, where q and θ_q are the radial and angular parameters of $\mathbf{q}$, then a rotation by α will transform $f_m(\mathbf{q})$ into a multiple of itself: $\mathbb{T}_\alpha f_m(\mathbf{q}) = \exp(im\alpha)f_m(\mathbf{q})$. As $\mathbb{T}_{2\pi} = \mathbb{1}$, m can only be an integer. The rotation-invariant self-adjoint operator $-i\,\partial/\partial\theta_q$ can be used to label $f_m(\mathbf{q})$, and these, we know from the theory of Fourier series, are orthogonal and complete in their inner product on $\theta_q \in (-\pi, \pi]$. Every function $f_m(\mathbf{q})$ (fixed m) will thus have the same, definite, rotation property *and so will its Fourier transform*. Consider now the two-dimensional Fourier transform of the set $\{f_m(\mathbf{q})\}$, $\{\tilde{f}_m(\mathbf{p})\}$ for fixed m and $\mathbf{f} \in \mathscr{L}^2(\mathscr{R}^2)$. Using the Bessel generating function (B.4) for $t = i\exp(i\theta)$, we can write the $\mathbb{F}_{(2)}$ kernel function as

$$\exp(\pm i\mathbf{p}\cdot\mathbf{q}) = \exp(\pm ipq\cos\theta) = \sum_{n=-\infty}^{\infty} (\pm i)^n J_n(pq)\exp(in\theta). \tag{8.78}$$

If $\mathbf{q}$ and $\mathbf{p}$ have polar coordinates q, θ_q and p, θ_p, $\mathbf{p}\cdot\mathbf{q} = pq\cos(\theta_p - \theta_q)$, and thus, for $\theta := \theta_p - \theta_q$, we can use the expansion (8.78) in finding

$$\begin{aligned}\tilde{f}_m(\mathbf{p}) &= \tilde{f}(p)\exp(im\theta_p)\\ &= (2\pi)^{-1}\int_0^\infty q\,dq\int_0^{2\pi} d\theta_q f(q)\exp(im\theta_q)\\ &\qquad\times\sum_{n\in\mathscr{Z}}(-i)^n J_n(pq)\exp[in(\theta_p-\theta_q)]\\ &= \int_0^\infty q\,dq f(q)\sum_{n\in\mathscr{Z}}(-i)^n J_n(pq)\exp(in\theta_p)\\ &\qquad\times(2\pi)^{-1}\int_0^{2\pi} d\theta_q\exp[i(m-n)\theta_q]\\ &= \int_0^\infty q\,dq f(q)(-i)^m J_m(pq)\exp(im\theta_p).\end{aligned} \tag{8.79}$$

That is,

$$\tilde{f}(p) = (-i)^m \int_0^\infty q\, dq f(q) J_m(pq), \tag{8.80a}$$

$$f(q) = i^m \int_0^\infty p\, dp \tilde{f}(p) J_m(pq) \tag{8.80b}$$

relate the "radial parts" of $f_m(\mathbf{q})$ and its $\mathbb{F}_{(2)}$ transform $\tilde{f}_m(\mathbf{p})$. For $m = 0$ (*invariance* under rotations) we recover the Hankel–Bochner transform (8.66) (of degree zero) for $N = 2$. For $m \neq 0$ (8.80) yields one transform for every m which differs from (8.66) for $m = N/2 - 1$ only in the powers of p and q in the integrand.

In N-dimensional space, the same kind of conclusion follows, except that the analogue of (8.78) is

$$\exp(\pm i\mathbf{p}\cdot\mathbf{q}) = (2\pi)^{N/2}(pq)^{1-N/2} \sum_{k=0}^{\infty} \exp(\pm i\pi k/2) \times J_{N/2+k-1}(pq) \sum_M Y_k^M(\Omega_q)^* Y_k^M(\Omega_p), \tag{8.81}$$

where Ω_q and Ω_p are the collective labels for the angular variables of $\mathbf{q}$ and $\mathbf{p}$ in Eqs. (8.60) and $Y_k^M(\Omega)$ are the *spherical harmonics* of rank k in N-space, M being a collective label for $N > 3$. [See, for instance, the book by Vilenkin (1968, Chapters IV and IX).] In N-space, definite transformation properties mean that we are dealing with functions of the kind $f_k^M(\mathbf{q}) = f(q) Y_k^M(\Omega_q)$. The spherical harmonics are orthogonal and complete on the space $\mathscr{L}^2(S_{N-1})$, so an analogue of the reduction (8.79) leads to

$$\tilde{f}(p) = p^{1-N/2} \exp(-i\pi k/2) \int_0^\infty q^{N/2}\, dq f(q) J_{N/2+k-1}(pq), \tag{8.82a}$$

$$f(q) = q^{1-N/2} \exp(i\pi k/2) \int_0^\infty p^{N/2}\, dp \tilde{f}(p) J_{N/2+k-1}(pq) \tag{8.82b}$$

for the radial parts of $f_k^M(\mathbf{q})$ and its $\mathbb{F}_{(N)}$ transform. Again, for $k = 0$ (*invariance*), we recover the Hankel–Bochner transform (8.66).

For the same value of the Bessel function index, (8.80), (8.82), and (8.66) differ only by powers of q and p. These can be easily absorbed into the definition of the function to be transformed. It has thus been found convenient to abstract the transform of the radial part from the number of dimensions of the original space and define *the Hankel* transform pair of order μ as

$$(\mathbb{H}_\mu \mathbf{f})(p) := f^{H\mu}(p) = \int_0^\infty dq f(q)(pq)^{1/2} J_\mu(pq), \tag{8.83a}$$

$$(\mathbb{H}_\mu^{-1} \mathbf{f}^{H\mu})(q) := f(q) = \int_0^\infty dp f^{H\mu}(p)(pq)^{1/2} J_\mu(pq). \tag{8.83b}$$

This is the form appearing in the Oberhettinger tables (1973) and has the advantage of symmetry in having the kernel a function of pq only.

Most authors call (8.83) *the* Hankel transform, while our original pair (8.66) is referred to as the Bochner transform. The Hankel transform occupies a part of the books by Sneddon (1951, Chapter 2; 1972, Chapter 5). For further material on this and related transforms, the reader is referred to the specialized literature. On convolution there are articles by Griffith (1957, 1958) and Haimo (1965); the latter deals in detail with applications. Extensive tables of Hankel transforms can be found in the Bateman manuscript project (Erdelyi *et al.*, 1954, Chapter VIII) and the tables by Oberhettinger (1973).

8.4.7. Other Integral Transforms with Cylindrical Function Kernels

Neumann transforms of order μ replace the Bessel function kernel in the Hankel transform by a Neumann function $(pq)^{1/2}N_\nu(pq)$. [See Griffith (1958) and the Bateman manuscript project (Erdelyi *et al.*, 1954, Chapters IX and XI) for Y and $\mathbf{H}$ transforms.] The inverse transform contains a *Struve* function kernel $(pq)^{1/2}\mathbf{H}_\mu(pq)$. A generalization of these involving *Lommel* functions can be found in the Oberhettinger tables (1973, Chapter VI).

Weber transforms of order μ are defined when

$$q[J_\mu(pq)N_\mu(pa) - J_\mu(pa)N_\mu(pq)],$$

the annular membrane determinant function in Eq. (6.37), is used as an integration kernel on (a, ∞). The inverse transform divides the direct kernel by $J_\mu(ap)^2 + N_\mu(ap)^2$ and integrates p on $(0, \infty)$. See the original paper by Titchmarsh (1923) and one by Griffith (1956).

The *Meijer–Bessel* or *Meijer K* transform of order μ makes use of the kernel $(pq)^{1/2}K_\mu(pq)$ containing the *Macdonald* function. The inverse transform integrates with a modified Bessel function $(pq)^{1/2}I_\mu(pq)$ over a Bromwich contour. Several Indian mathematicians have published articles on this subject (Verma, 1959; Saxena, 1959; and Sharma, 1963, 1965). The Bateman manuscript project [Erdelyi *et al.* (1954)] devotes Chapter X to giving a table of these. This is actually a particular case of the *Meijer transform*, introduced by Meijer (1940) as a generalization of the Laplace transform whose transform kernels are $\exp(\mp pq)/2\cdot(pq)^{\mp k-1/2}$ times the *Whittaker* functions $W_{k-1/2,m}(pq)$ for the direct and $M_{k-1/2,m}(pq)$ for the inverse transform. Both k and m are free parameters. Vilenkin (1968, Chapter VIII) gives many group-theoretical and special-function relations for integrals with Whittaker function kernels.

Kontorovich and Lebedev (1938) introduced a particular integral transform for the solution of problems in diffraction. It involves as a transform kernel a Macdonald function of imaginary index $K_{iq}(p)$ over $q \in (0, \infty)$

and for the inverse transform the kernel $2\pi^{-2}\sinh(\pi q)K_{iq}(p)/p$, also over $p \in (0, \infty)$. Its relation with some of the Neumann series (Section 6.4) is akin to the relation of the Mellin transform with the Taylor series in Fig. 8.6. The conditions for validity of the transform pair were further explored by Lebedev (1947), and the transform was generalized (Lebedev, 1949a, 1949b). A transform table appears in the Bateman manuscript project (Erdelyi *et al.*, 1954, Chapter XII) and in Oberhettinger (1973, Chapter VI). Sneddon (1972, Chapter 6) treats this transform in some detail and applies it to the study of harmonic functions in cylindrical coordinates.

8.5. Other Integral Transforms

For the most part, integral transforms can be seen as the continuous analogue of series expansions. The underlying unity is that the expanding functions in the series and the integral kernel in transforms are usually eigenfunctions of a given operator, self-adjoint in some domain. In this section, after some rather soft-focus remarks on the Sturm–Liouville point of view, we shall examine a few examples as well as other transforms which, unnamed, have appeared before or which are common in the current literature.

8.5.1. The Sturm–Liouville Problem and Integral Transforms

Assume $\mathbb{H}$ is an operator which is self-adjoint in the (Hilbert) space of functions $\mathscr{L}^2(\mathscr{Q})$, where $\mathscr{Q} \subseteq \mathscr{R}$ with some properly chosen boundary conditions. Assume, further, that we know its eigenfunctions, labeled uniquely by a (possibly collective) index $p \in \mathscr{P} \subseteq (\mathscr{R}, \mathscr{Z})$,

$$\mathbb{H}\Psi_p(q) = \lambda(p)\Psi_p(q), \qquad q \in \mathscr{Q}, \tag{8.84}$$

and its spectrum $\mathscr{S} = \lambda(\mathscr{P}) \subseteq \mathscr{R}$. The set of functions $\{\Psi_p(q)\}_{p\in\mathscr{P}}$ can be shown under certain restrictions to constitute a generalized (Dirac) basis, orthogonal and complete for $\mathscr{L}^2(\mathscr{Q})$. This is the generalized Sturm–Liouville problem, similar to the one sketched in Section 6.4. The spectrum, being a continuous set, however, is indicative of a considerably more delicate mathematical theory. The overall (simplified) features are not too difficult to state roughly: the orthogonal basis functions can be normalized so that Dirac orthonormality holds,

$$(\boldsymbol{\Psi}_p, \boldsymbol{\Psi}_{p'}) = \int_{\mathscr{Q}} dq\Psi_p(q)^*\Psi_{p'}(q) = \delta(p - p'), \tag{8.85a}$$

and completeness holds,

$$\int_{\mathscr{P}} dp\Psi_p(q)^*\Psi_p(q') = \delta(q - q'). \tag{8.85b}$$

Although (8.85a) tells us that the $\Psi_p(q)$ do *not* belong to $\mathscr{L}^2(\mathscr{Q})$, they do nevertheless form a generalized *basis* for that space so that for any $f(q) \in \mathscr{L}^2(\mathscr{Q})$ we can define its *transform* function,

$$f^T(p) := \int_{\mathscr{Q}} dq \Psi_p(q)^* f(q), \tag{8.86a}$$

and be assured that the inverse transform or synthesis reproduces (generally *in the norm*) the original function as

$$f(q) = \int_{\mathscr{P}} dp \Psi_p(q) f^T(p). \tag{8.86b}$$

This suggests seeing the integral transform—*passive* point of view; recall Section 1.3—as a change of basis, where $f(q)$ and $f^T(p)$ are the coordinates of the same vector $\mathbf{f} \in \mathscr{L}^2(\mathscr{Q})$ in two bases, the latter in the $\{\mathbf{\Psi}_p\}_{p\in\mathscr{P}}$-basis as $(\mathbf{\Psi}_p, \mathbf{f}) = f^T(p)$ and the former in the basis of Dirac δ's, $\{\boldsymbol{\delta}_q\}_{q\in\mathscr{Q}}$, where $(\boldsymbol{\delta}_q, \mathbf{f}) = f(q)$. Equation (8.86b) can be formally "proven" by multiplying (8.86a) by $\Psi_p(q')$, integrating over $p \in \mathscr{P}$, exchanging integrals, and using (8.85b). Equivalently, multiplication of (8.86b) by $\Psi_{p'}(q)$, integration over $q \in \mathscr{Q}$, and use of (8.85a) yield (8.86a). As a consequence of (8.84)–(8.86), the generalized Parseval relation,

$$(\mathbf{f}, \mathbf{g})_{\mathscr{Q}} = \int_{\mathscr{Q}} dq f(q)^* g(q) = \int_{\mathscr{P}} dp f^T(p)^* g^T(p), \tag{8.86c}$$

will also hold.

8.5.2. Fourier, Mellin, and Repulsive Oscillator Transforms

The Fourier transform (7.1) can be seen as stemming from the eigenbasis expansion of a defining operator $\mathbb{P} := -id/dq$, self-adjoint on $\mathscr{Q} = \mathscr{R}$ [see Eqs. (7.55) and (7.56)]. Its eigenfunctions are $(2\pi)^{-1/2}\exp(ipq)$, $p \in \mathscr{P} = \mathscr{R}$, and the spectrum is $\mathscr{S} = \mathscr{R}$. The set is an orthogonal and complete basis for $\mathscr{L}^2(\mathscr{R})$. The defining operator can also be taken to be $\mathbb{P}^2/2$, whose eigenfunctions are the same as above but whose spectrum is $\mathscr{S} = \mathscr{R}^+$ *twice* [as $\mathscr{P} = (\mathscr{R}^+, \pm)$]. The latter has the advantage of defining, equivalently, the sine and cosine Fourier transforms: These are eigenfunctions of $\mathbb{P}^2/2$ but not of $\mathbb{P}$.

The Mellin bilateral transform (8.26) can be built by looking for the eigenfunctions of the operator $\frac{1}{2}(\mathbb{Q}\mathbb{P} + \mathbb{P}\mathbb{Q})$, namely, $(2\pi)^{-1/2}q_\pm^{ip-1/2}$. Here $\mathscr{S} = \mathscr{R}$, but $\mathscr{P} = (\mathscr{R}, \pm)$, i.e., the spectrum covers $\mathscr{R}$ *twice*. Orthogonality and completeness (8.86) are given the forms (8.27) and (8.28).

We also have the transform defined by the repulsive oscillator Schrödinger Hamiltonian $\frac{1}{2}(\mathbb{P}^2 - \mathbb{Q}^2)$, which is closely related to the Mellin transform. The eigenfunctions of this operator are the $\chi_p{}^{\pm}(q)$ found in (7.203).

They can serve to define a *repulsive* oscillator transform with the characteristics (8.84)–(8.86).

Exercise 8.30. Show that if the defining operator is $\mathbb{P}^2 - (\mu^2 - \frac{1}{4})\mathbb{Q}^{-2}$, self-adjoint on $\mathscr{L}^2(\mathscr{R}^+)$, the resulting transform is the Hankel transform of order μ given by (8.83).

8.5.3. Airy Transforms

Usually, a solid link with the Fourier transform—for which the eigenbasis properties are well established—will prove the orthogonality and completeness for a given transform basis function set. This was the path we followed for Mellin, repulsive oscillator, and Hankel transforms.

One more transform can easily be presented by this method. Consider the operator and corresponding eigenvalue equation

$$\mathbb{H}^l \Psi_\lambda^{\,l}(q) := (\tfrac{1}{2}\mathbb{P}^2 + \mathbb{Q})\Psi_\lambda^{\,l}(q) = \lambda \Psi_\lambda^{\,l}(q). \tag{8.87}$$

This equation happens to be the (time-independent) Schrödinger equation for the free-fall or linear potential. It was solved for $\lambda = 0$, in (7.61)–(7.64), in terms of the Airy function. Actually that is almost all we need since $\Psi_\lambda^{\,l}(q) = \Psi_0^{\,l}(q - \lambda)$ is the solution of (8.87) in terms of the $\lambda = 0$ solution, as can be ascertained by collecting all terms on the left-hand side and changing variables. We can thus write the solution to (8.87) in terms of (7.64) with a translated argument, viz.,

$$\Psi_\lambda^{\,l}(q) = 2^{1/3}\,\mathrm{Ai}[2^{1/3}(q - \lambda)] \tag{8.88}$$

[having chosen $c = (2\pi)^{-1/2}$]. Moreover, we can easily show that the set (8.88), for $\lambda \in \mathscr{R}$, is orthogonal and complete. Indeed, the Fourier transform of (8.88) is given by (7.63), multiplied by an exponential factor due to translation [Eq. (7.28)]:

$$\tilde{\Psi}_\lambda^{\,l}(p) = (2\pi)^{-1/2}\exp(-i\lambda p)\exp(ip^3/6). \tag{8.89}$$

Now, *this* set of functions *is* orthogonal and complete for $p \in \mathscr{R}$ and $\lambda \in \mathscr{R}$. The last λ-independent exponential factor does not alter this property, as can be shown by an argument parallel to that leading from the completeness of the bilateral Mellin basis to the completeness of the repulsive oscillator wave functions in Section 8.2. The inverse Fourier transform of (8.89), namely (8.88), will thus have the claimed property. Equation (8.88) defines the integral kernel of a transform which we can call *Airy's* transform.

An integral transform (8.84)–(8.86), in the *active* point of view (recall Section 1.3), is quite obviously associated with a *linear operator*—that is, if $f^T(p)$ and $g^T(p)$ are the transforms of $f(q)$ and $g(q)$, then $af^T(p) + bg^T(p)$ will be the transform of $af(q) + bg(q)$ for $a, b \in \mathscr{C}$—and thus we can define a linear operator $(\mathbb{T}\mathbf{f})(p) := f^T(p)$.

Most of the transforms we have examined thus far are *unitary* (those of Fourier, bilateral Mellin, Hankel, repulsive oscillator, and Airy but *not* those of Laplace or ordinary Mellin). As the corresponding Parseval identities suggest, the mapping afforded by $\mathbb{T}$ is *isometric.* The fact that $\mathscr{L}^2(\mathscr{R})$ can be shown to be mapped onto itself under Fourier and Airy transforms makes the transform operators *unitary* [since $\mathscr{L}^2(\mathscr{R})$ is a *Hilbert* space]. The Hankel transforms achieve the same for the (Hilbert) space $\mathscr{L}^2(\mathscr{R}^+)$. The bilateral Mellin and repulsive oscillator transforms are also unitary, although they map $\mathscr{L}^2(\mathscr{R})$ onto $\mathscr{L}_+{}^2(\mathscr{R}) \oplus \mathscr{L}_-{}^2(\mathscr{R})$ for the two values of the dichotomic index. Finally, the harmonic oscillator functions also provide a unitary mapping (7.180) of $\mathscr{L}^2(\mathscr{R})$ onto l^2, the (Hilbert) space of square-summable sequences.

8.5.4. Gauss–Weierstrass Transforms

Not all integral transforms are unitary though. When we look at the time evolution of systems governed by linear differential equations, linear mappings of functions through integral kernels become abundant. Consider the simple heat diffusion in one dimension described by the Green's function in (8.58) with initial conditions $f(q)$ at time $t = 0$ and $a^2 = 1/2$. Its time evolution is given by

$$(\mathbb{G}_t^h \mathbf{f})(q) := f^{G(t)}(q) = (2\pi t)^{-1/2} \int_{-\infty}^{\infty} dq' f(q') \exp[-(q - q')^2/2t]. \quad (8.90)$$

This is a linear mapping of a large function space [containing $\mathscr{L}^2(\mathscr{R})$] *into* $\mathscr{C}_\downarrow{}^\infty$, which has been called the *Gauss* or *Weierstrass* transform at time t. Although the "total heat" $\int dq f^{G(t)}(q)$ is constant, the usual inner product $(\mathbf{f}, \mathbf{f})$ is *not*. Hence $\mathbb{G}_t^h$ is not a unitary transform in the usual sense. Nevertheless, in Part IV we shall see that if an appropriate inner product is given, (8.90) *can* be turned into a unitary transform. The transform (8.90) and its *inversion* have an important bearing on the theory of heat diffusion. This was initially studied by Doetsch (1936) and Tricomi (1936, 1938). Since then, it has been the subject of several articles by Hartmann and Wintner (1950), Blackman (1952), Widder (1956, 1964), Rooney (1957, 1958, 1963), Bilodeau (1961), and Nessel (1965). There is one recent book on the heat equation by Widder (1975).

8.5.5. Complex Extensions and Analytic Continuations

A subject which will be more extensively developed in Part IV is the set of integral transforms obtained from the time evolution of *four* types of Schrödinger equations: (a) the harmonic oscillator, (b) the free particle, (c) the repulsive oscillator, and (d) the linear potential. The Green's functions for these cases all have the general form $\exp[i(Aq^2 + Bqp + Cp^2)]$ for A, B, and C complex. This will define (the semigroup of) *complex canonical transforms*. They are all unitary in the appropriate Hilbert spaces.

Wave equations describing diffusive–elastic systems also provide integral transforms on *pairs* of functions representing elongation and velocity. They can be made unitary (thus far) only in the case when no diffusion is present. The inner product to appear in the Parseval identity is the sesquilinear form associated with the total energy of the system.

Integral transforms between pairs of functions can arise also as *analytic continuations* of series. This rather cryptic remark applies to the case of the *Mehler–Fok* transform, which can be seen as a Sturm–Liouville problem or as an analytically continued version of the Legendre transform mentioned in Section 6.4. The latter expands functions as series of Legendre polynomials $P_n(x)$. By complex contour integration techniques (usually referred to as the *Sommerfeld–Watson transform*), the series sum is replaced by an *integral* with a kernel $(2\nu + 1)P_\nu(x)/\sin \pi\nu$ over ν along a vertical path in the complex ν-plane at $\rho + i\sigma$ for fixed $\rho > -\frac{1}{2}$ and over $\sigma \in \mathscr{R}$ as shown in Fig. 8.6. This transform is used in high-energy elementary particle physics for relativistic scattering amplitude expansions in the direct and crossed channel [see the review article by Kalnins *et al.* (1975, Section III-B and the references within)]. Application of this transform to the diffraction and reflection by a wedge has been made by Oberhettinger (1954, 1958). This transform has a family of group-theoretical generalizations related by relativistic partial-wave expansions. They have been amply discussed by Vilenkin (1968, Chapter X). Sneddon (1972) dedicates Chapter 7 in his book to the study of the Mehler–Fok transform and its applications.

8.5.6. Hilbert Transforms

Sturm–Liouville theory need not be involved in all transforms. In Section 7.4 we saw that the real and imaginary parts of the Fourier transform of a causal function were related by (7.146) (for $a = 0$) as

$$f_R(p) = \pi^{-1}\mathscr{P}\int_{-\infty}^{\infty} dp'(p - p')^{-1}f_I(p'), \tag{8.91a}$$

$$f_I(p) = -\pi^{-1}\mathscr{P}\int_{-\infty}^{\infty} dp'(p - p')^{-1}f_R(p'), \tag{8.91b}$$

where $\mathscr{P}$ stands for the integral's principal value and the tildes have been dropped. Equations (8.91) define $f_I(p)$ as the *Hilbert* transform of $f_R(p)$. As the definition of the Hilbert transform is closely related to analyticity, it has served, for instance, in constructing a generalized *phasor* formalism as for alternating-current theory, which is applicable to general frequency-modulated signals. A given real signal $\sigma(t)$ and its Hilbert transform $\tau(t)$ are merged into an *analytic* complex signal $\sigma(t) + i\tau(t) =: \rho(t)\exp[i\omega(t)t]$, where $\rho(t)$ is the signal *envelope* and $\omega(t)$ the *instantaneous frequency*. As an example, we can recall the repulsive oscillator wave functions $\chi_\lambda{}^{\pm}(q)$ in Eqs. (7.203), shown in Fig. 7.11. As $\chi_\lambda{}^{+}(q)$ is the inverse Fourier transform of a function having support on the positive half-axis, it follows that $\operatorname{Im}\chi_\lambda{}^{+}(q)$ is the Hilbert transform of $\operatorname{Re}\chi_\lambda{}^{+}(q)$. Figure 7.11 shows the envelope of the former to be $|\chi_\lambda{}^{+}(q)|$. A sound mathematical treatment of the Hilbert transform can be seen in Titchmarsh's Fourier classic (1937, Chapter V) or, if available, in Cotler's dissertation (1953). Further work on the application of the Hilbert transform to the theory of causal filters can be seen in the article by Urkowitz (1962) and the books by Bracewell (1965, Chapter 11) and Sneddon (1972, Section 3-21). Tables of Hilbert transforms can be found in the Bateman manuscript project (Erdelyi *et al.*, 1954, Chapter XV).

8.5.7. Stieltjes Transforms

The *Stieltjes* transform is defined as the square of the unilateral Laplace transform:

$$f^s(q) = (\mathbb{L}^2\mathbf{f})(q) = \int_0^\infty dq''\exp(-qq'')\int_0^\infty dq' f(q')\exp(-q''q')$$

$$= \int_0^\infty dq'(q + q')^{-1}f(q'). \tag{8.92a}$$

The original function is regained as

$$f(q) = (2\pi i)^{-1}\lim_{\varepsilon\to 0^+}[f^s(-q - i\varepsilon) - f^s(-q + i\varepsilon)], \tag{8.92b}$$

as can be ascertained by noting that (8.92a) is related to the Cauchy representation (7.136) by a change of sign in the argument and a factor of $2\pi i$. If $f(q)$ is continuous at q, (8.92b) follows from Eq. (7.137d). If $f(q)$ is discontinuous, one has to substitute as usual, $\lim_{\varepsilon\to 0^+}[f(q + \varepsilon) + f(q - \varepsilon)]/2$ for the left-hand side of (8.92b). The Stieltjes transform arose from the Stieljes moment problem (Titchmarsh, 1937, Section 11.9). It has been investigated thoroughly by Widder (1937, 1938) and occupies Chapter VIII of his 1941 book. Several generalizations of the Stieljes transform involve higher powers of the denominator in (8.92a) [in Widder's book (1941)], a hypergeometric function (Varma, 1951), or a Whittaker function (Arya, 1963). Tables of Stieltjes transforms can be found in the Bateman manuscript project (Erdelyi *et al.*, 1954, Chapter XIV).

8.5.8. Convolution Transforms

Integral transforms or various general types have been further considered in the literature. One class involves the *convolution transform*, which is of the general form

$$f^G(p) = \int_{-\infty}^{\infty} dq f(q) G(p - q), \tag{8.93}$$

where G is a rather general function including, for instance, the diffusion transform kernel. Various properties of the construct (8.93), the possibility of inversion, and its relation to hyperdifferential operators have been the subject of the book by Hirschmann and Widder (1955). Browsing through the list of references in Widder's books, one discovers many other transforms associated with as many other names. It will serve us to close the list here and reserve Part IV for the presentation of canonical transforms.

and (c), as do the solutions of the wave equation in other dimensions. As to property (d), the behavior is different. The disturbance, if originally localized, will develop a trailing wake, because of the non-δ form of $G_2{}^w(q, t)$, which *smears* any initial condition out of its original sharpness. This trailing wake, backwave or reverberation is a characteristic of all even-dimensional spaces. A stone thrown in a pond does not quite reproduce, thus, the behavior of waves in three-dimensional space.

8.4.5. General Solution of the Diffusive–Elastic Equation

For the Green's function of the N-dimensional general elastic–diffusive medium, we have to calculate, as in (8.70), the inverse Hankel transform of (8.49), namely,

$$\begin{aligned} G_N(\mathbf{q}, t) &= \big(\mathbb{F}_{(N)}\tilde{G}(\cdot, t)\big)(\mathbf{q}) = \big(\mathbb{H}^B_{N/2-1}\tilde{G}(\cdot, t)\big)(q) \\ &= c^2 q^{-N/2+1} \exp(-\Gamma t) \int_0^\infty dp\, p^{N/2}(c^2p^2 - \Gamma^2)^{1/2} \\ &\quad \times \sin[t(c^2p^2 - \Gamma^2)^{1/2}] J_{N/2-1}(pq). \end{aligned} \tag{8.75}$$

This is a rather difficult integral to do "by hand." It appears in the literature, however, as a particular case of the discontinuous Sonine integrals. [See Watson (1922, Section 13.47). In the tables of Hankel transforms by Oberhettinger (1973), it can be found by Eq. 6.43-II.] The result is

$$\begin{aligned} G_N(q, t) &= c(\pi/2)^{1/2}[\Gamma c^{-1}(c^2t^2 - q^2)^{-1/2}]^{(N-1)/2} \\ &\quad \times \exp(-\Gamma t) I_{-(N-1)/2}\big(\Gamma c^{-1}(c^2t^2 - q^2)^{1/2}\big)\Theta(ct - q), \\ &\qquad q \neq ct, \end{aligned} \tag{8.76}$$

where $I_\nu(z)$ is the *modified Bessel function* (see Appendix B) and Θ the usual Heaviside step. This function has been plotted for various values of parameters and variables in Fig. 8.10. The Green's function (8.76) includes δ terms at $q = ct$ for $N > 2$. These can be obtained for odd N using (8.68) and the fact that $G_1(q, t)$ is simply discontinuous at the advancing edge of the wave. For N even, one starts from $G_2(q, t)$.

Exercise 8.27. Consider the diffusion equation limit $c \to \infty$ of the Green's function (8.76), recalling that $I_\nu(z) \sim (2\pi z)^{-1/2} \exp(z)$ as $z \to \infty$. Verify that Eq. (8.58) is correctly reproduced.

Exercise 8.28. Consider the wave equation limit $a \to \infty$ of (8.76) using $I_\nu(z) \sim (z/2)^\nu/\Gamma(\nu + 1)$ for $\nu \neq -1, -2, \ldots$ as $z \to 0$. This verifies only the even-dimensional cases. Show that for $N = 2$, (8.73) is correctly given. For N odd and larger than 1, the result is zero since (8.76) holds for $q \neq ct$.